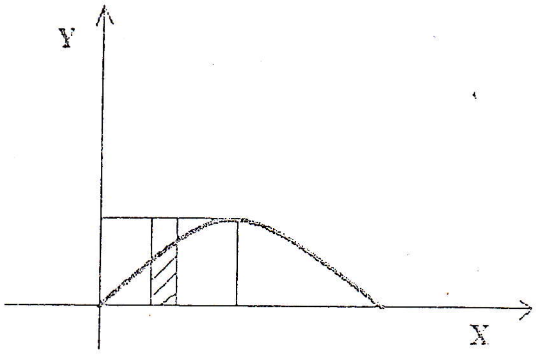


$\Delta V = 2\pi x f(x)\Delta x$

$V = 2\pi\int_0^{\pi} x\sin x\, dx = 2\pi[\sin x - x\cos x]_0^{\pi} = 2\pi^2$

03. a) A student asked to calculate $\lim_{x\to 0}\frac{\sin x}{x+x^2}$ applies Hospital's rule twice in succession and finds $\lim_{x\to 0}\frac{\sin x}{x+x^2} = \lim_{x\to 0}\frac{\cos x}{1+2x} = \lim_{x\to 0}\frac{-\sin x}{2} = 0$. Explain why this answer is wrong and find the correct answer.

b) Evaluate $\lim_{x\to\frac{\pi}{2}}\frac{\sec x}{\tan x}$ 3.5 marks

Answer:

a) The answer is wrong because $\frac{\cos x}{1+2x}$ is not the indeterminate form when $x \to 0$ and consequently the hospital's rule is not applied

The correct calculation is $\lim_{x\to 0}\frac{\sin x}{x+x^2} = \lim_{x\to 0}\frac{\cos x}{1+2x} = 1$

b) $\lim_{x\to\frac{\pi}{2}}\frac{\sec x}{\tan x} = \frac{\infty}{\infty} I.F$

$\lim_{x\to\frac{\pi}{2}}\frac{\frac{1}{\cos x}}{\frac{\sin x}{\cos x}} = \lim_{x\to\frac{\pi}{2}}\frac{1}{\sin x} = 1$

04. Let C and D be events defined on the same sample space such that $p(C) = \frac{4}{7}$, $p(C\cap\overline{D}) = \frac{1}{3}$; $p(C/D) = \frac{5}{14}$ with $\overline{D}$ the complementary event of D.

a) Find $p(C\cap D)$, $p(D)$ and $p(D/C)$.

b) Explain whether C and D are independent events. 4.5 marks

Answer:

a) $C = (D\cap\overline{D})\cup(C\cap D)$ and $(C\cap\overline{D})\cap(C\cap D) = \Phi$

$P(C) = P(C\cap\overline{D}) + P(C\cap D) \Rightarrow P(C\cap D) = P(C) - P(C\cap\overline{D})$

$P(D/C) = \frac{P(C\cap D)}{P(C)} = \frac{5}{21} x \frac{7}{4} = \frac{5}{12}$

$P(D) = \frac{P(C\cap D)}{P(C/D)} = \frac{5}{21} x \frac{14}{5} = \frac{2}{3}$

b) $P(C/D) \neq P(D)$ or $P(C\cap D) \neq P(C).P(D) \Rightarrow$ *C and D are not independent.*

05. In a certain college: 65% of the students are boarder, 55% of the students are female and 35% of the students are male boarder. Find the probability that a student chosen at random from all the students in the college is

a) a day student;

b) female day student. 3 marks

Answer:

a) Let the element I be " boarder student"

F be " girl student"

we have P(I) = 0.65 and P(F) = 0.55

$P(\bar{I}) = 1 - P(I) = 1 - 0.65 = 0.35$

b) $P(F \cap \bar{I}) = ?$

We know that $I = (F \cap I) \cup (\bar{I} \cap \bar{F})$ and that $P(I \cap F) \cap (I \cap \bar{F}) = \Phi$

Thus $P(\bar{I} \cap F) = P(I) - P(I \cap F)$

As the same $P(I \cap F) = P(I) - P(I \cap \bar{F}) = 0.65 - 0.35 = 0.30$

Thus $P(\bar{I} \cap F) = P(F) - P(I \cap F) = 0.55 - 0.30 = 0.25$

06. In Euclidian space $(\Re, +, .)$ find

a) Parametric equations of the line of intersection of the planes 2x+y+z = 4 and 3x-y+z =3.

b) an equation for the plane that passes through the point P(1, 3, -2) and contains the line of intersection of the planes x-y+z = 1 and x+y-z = 1. 4 marks

Answer:

a) Let D be the line of intersection, we have:

$$D \equiv \begin{cases} 2x + y + z = 4 \\ 3x - y + z = 3 \end{cases} \Leftrightarrow D \equiv \begin{cases} y = \frac{1}{2} + \frac{1}{2}x \\ z = \frac{7}{2} - \frac{5}{2}x \end{cases}$$

Let pose x = λ (the parameter)

$$D \equiv \begin{cases} x = \lambda \\ y = \frac{1}{2} + \frac{1}{2}\lambda \text{(parametric equations)} \\ z = \frac{7}{2} - \frac{5}{2}\lambda \end{cases}$$

N.B: There are different forms of answers, it depends on the variable taken as parameter.

b) Let find the Cartesian equations of the plan which contain the line D' and passes through the point P(1, 3, -2) with $D' \equiv \begin{cases} x - y + z = 1 \\ x + y - z = 1 \end{cases}$

Thus $P_o(1,1,1)$ and P(1, 0, 0) (many answers: y = z can take any values of $\Re$)

The vectors $\overline{PP_o}(0, -2, 3)$ and $\overline{PP_1}(0, -3, 2)$ there are said "director vectors of the asked plan

Thus the equation $\begin{bmatrix} x - 1 & y - 3 & z + 2 \\ 0 & -2 & 3 \\ 0 & -3 & 2 \end{bmatrix} = 0$

Let x =1 or x-1 =0

07. Let $z = -12i\sqrt{3}+12$, $t = -6\sqrt{3} + 6i$ and $w = \frac{z}{t}$ be complex numbers.

a) Compute the modulus and an argument of z and t.

b) Express w and w^6 in their polar and standard forms. 4 marks

Answer:

a) Module of Z is $|z| = (12\sqrt{1+3}) = 24$ and module of t is $|t| = (6\sqrt{1+3}) = 12$

Argument of Z = $\begin{cases} \cos\theta = \frac{1}{2} \\ \sin\theta = -\frac{\sqrt{3}}{2} \end{cases} \Rightarrow \theta = \frac{5\pi}{3} + 2k\pi, k\epsilon\Re$

Argument of t = $\begin{cases} \cos\theta = -\frac{\sqrt{3}}{2} \\ \sin\theta = \frac{1}{2} \end{cases} \Rightarrow \theta = \frac{5\pi}{6} + 2k\pi, k\epsilon\Re$

Where w = $\frac{24}{12}\left(\cos\left(\frac{5\pi}{3} - \frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{3} - \frac{5\pi}{6}\right)\right) = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$ (trigo. form)

$= -\sqrt{3} + i$ (algebraic form)

b) $w^6 = 2^6\left(\cos 6\frac{5\pi}{6} + i\sin 6\frac{5\pi}{6}\right) = 2^6(\cos 5\pi + i\sin(-5\pi)) = -2^6 = -64$

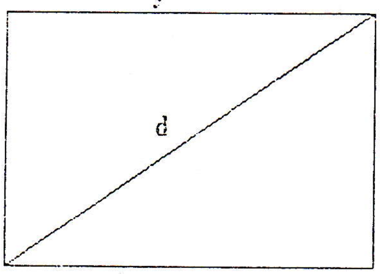
08. The fundamental theorem of calculus seems to say that $\int_{-1}^{1} \frac{1}{x^2}dx = \left[-\frac{1}{x}\right]_1^1 = -2$ in apparent contradiction to the fact that $\frac{1}{x^2}$ is always positive. What is wrong here? 2 marks

Answer:

The function to integrate is $f(x) = \frac{1}{x^2}$ is not bounded when $x \to 0$ and however is not continuous under $[-1, 1]$ thus the fundamental theorem is not applicable in this case.

09. Find the maximum possible area of a rectangle with diagonal of length 16m. Find all real numbers x such that $4^x - 2^x - 2 \geq 0$ 4.5 marks

Answer:



$d^2 = x^2 + y^2 \Rightarrow y^2 = d^2 - x^2 \Rightarrow y^2 = 16 - x^2$

$A(x) = xy = x\sqrt{16^2 - x^2}$ with $0 < x \leq 16$

Using derivative, $A(x) = \frac{16^2 - 2x^2}{\sqrt{16^2 - x^2}}$

The area of rectangle is

X		(8, 2) ???????	
A'	++	0	- -
A	↗ 0	A(3, 2)	↘ 0

The maximum area is thus $A(8\sqrt{2}) = 128cm^2$ with $x = y = 8\sqrt{2}$ m

Let solve the equation $4^x - 2^x \geq 0$

$4^x - 2^x - 2 \Leftrightarrow (2^x - 2)(2^x + 1) \geq 0 \Leftrightarrow 2^r \leq -1$ or $2^x \geq 2.1$

As $2x > 0\ \forall \in \Re$ *we exclude* $2 \leq -1$

Thus 4x -2x -2 > 0 ⇔ $2^n > 2 \Leftrightarrow x > \frac{ln2}{ln2}$

$S = \left[\frac{ln2}{ln2}, +\infty\right[= [1, +\infty[$

10. If a diagonal of a polygon is defined to be a line joining any two non-adjacent vertices, how many diagonals are there in a polygon of n sides? 2.5 marks

Answer:

Number is $\frac{n}{2}(n-3)$

11. a) Find the angle θ between the planes with equations 2x+3y-z = -3 and 4x+5y+z = 1.
b) Then write symmetric equations of their line intersection L. 4 marks

Answer:

a) The vectors
$\vec{n}(2,3,-1)$ *and* $\vec{m} = (4,5,1)$ *are normal to the planes of equations:*
$2x + 3y - z = -3$ *and* $4x + 5y + z = 1$ respectively.

Thus $\cos\theta = \frac{\vec{n}\,\vec{m}}{\|\vec{n}\|.\|\vec{m}\|} = \frac{8+15-11}{\sqrt{14}\sqrt{42}} = \frac{11\sqrt{3}}{21}$

and $\theta = \arccos\left(\frac{11\sqrt{3}}{21}\right) \cong 24.81^0$

b) For finding the system of Cartesian equation of the line L, we must find two points: P_o and P_1 of L.
One point of L and the director vector of L (this is the product vector of $\vec{n}$ *and* $\vec{m}$)
Let use the first alternative: to find P_o and P_1 give the arbitrary value to any one of the variables x, y and we solve the obtained system:

Let x = 1 $\begin{cases} x = 1 \\ 3y - z = -5 \\ 5y + z = -3 \end{cases} \Leftrightarrow \begin{cases} x = 1 \\ y = -1 \\ z = 2 \end{cases} \Leftrightarrow P_o = (1, -1, 2)$

Let x = 1 $\begin{cases} x = 5 \\ 3y - z = -13 \\ 5y + z = -19 \end{cases} \Leftrightarrow \begin{cases} x = 5 \\ y = -4 \\ z = 1 \end{cases} \Leftrightarrow P_1 = (5, -4, 1)$

The director vector of L is $\overrightarrow{P_oP_1}(4,-3,-1)$ *and* $L \equiv \frac{x-1}{4} = \frac{y+1}{-3} = \frac{z-2}{-1}$

12. Suppose that it is known that 1-2i is a zero of the fourth-degree polynomial $f(x) = x^4 - 3x^3 + x^2 + 7x - 30$. Find all zeros of f(x). 4 marks

Answer:

Because complex zeros occur in conjugate pairs, you know that 1+2i is also a zero of f.

Both [x-(1-2i)] and [x-(1+2i)] are factors of f

$[x-(1-2i)][x-(1+2i)] = [(x-1)+2i][(x-1)-2i] = (x-1)^2 - 4i^2 = (x-1)(x-1) + 4 = x^2 - x - x + 1 + 4$
$= x^2 - 2x + 5$

$$x^4 - 3x^3 + x^2 + 7x - 30 \;\big|\; x^2 - 2x + 5$$
$$-x^4 + 2x^3 - 5x^2 \qquad\qquad x^2 - x - 6$$
$$0 \quad -x^3 - 4x^2 + 7x$$
$$+x^3 - 2x^2 + 5x$$
$$0 - 6x^2 + 12x - 30$$
$$+6x^2 - 12x + 30$$
$$0$$

$f(x) = (x^2 - 2x + 5)(x^2 - x - 6) = (x^2-2x+5)(x^2+2x-3x-6) = (x^2-2x+5)[(x(x+2)-3(x+2)]$
$= (x^2 - 2x + 5)(x+2)(x-3)$

The zeros of f are x = 1+2i, x = 1-2i, x = -2 and x = 3

$S = \{-2, 3, (1-2i), (1+2i)\}$

13. a) Determine whether the series $(U_n)_{n\in\mathbb{N}}$ given by $U_n = \frac{n+3}{4}$ is geometric or arithmetic.

b) Calculate $\sum_1^{20} U_n$. 3 marks

Answer:

a) $U_n = \frac{n+3}{4}$ and $U_{n+1} = \frac{n+4}{4} = \frac{n}{4} + 1$

$U_{n+1} - U_n = \frac{n}{4} + \frac{4}{4} - \frac{n}{4} - \frac{3}{4} = \frac{1}{4}$

$\Rightarrow (U_n)$ is arithmetic of common difference $r = \frac{1}{4}$

$$\sum_{n=0}^{20} U_n = U_1 + U_2 + \cdots + U_{20} = \frac{20}{2}(U_1 + U_{20}) = 10(2U_1 + 19r)$$

$$= 10\left(2.1 + 19.\frac{1}{4}\right) = \frac{135}{2}$$

14. The matrix $\begin{bmatrix} 1 & 2 & 0 \\ -1 & -4 & 0 \end{bmatrix}$ determines a linear application f: $\Re^3 \to \Re^2$ with respect to standard bases. Determine vector $\vec{u}(x, y, z)$ such that f(x, y, z) = (0, 0)

Answer:

$\text{Ker} f = \{(x, y, z) \in \Re^3 : f(x, y, z) = (0, 0)\}$

$$f(x, y, z) = (0, 0) \Leftrightarrow \begin{bmatrix} 1 & 2 & 0 \\ -1 & -4 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} x + 2y = 0 \\ -x - 4y = 0 \\ zq'q \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \\ z \in \Re \end{cases}$$

$\Rightarrow \text{Ker} f = \{(0, 0, z), z \in \Re\}$ (Dim ker f = 1

$\text{Im}\, f = \{(a, b) \in \Re^3 \exists\, (x, y, z) \in \Re^3 : f(x, y, z) = (a, b)\}$

$$f(x, y, z) = (a, b) \begin{cases} x + 2y = a \\ -x - 4y = b \\ zq'q \end{cases} \Rightarrow \begin{cases} x = a + 2b \\ y = -\frac{1}{2}(a + b) \\ z \in \Re \end{cases}$$

For each couple (a, b) of reals $\exists\, (x, y, z)$ of $\Re^3$ such that $f(x, y, z) = (a, b)$ i.e:

$\text{Im}\, f = \{(a, b) \in \Re^2\} = \Re^2$

N.B: To find Im f you can use the''dimension'' theorem (Grossman)

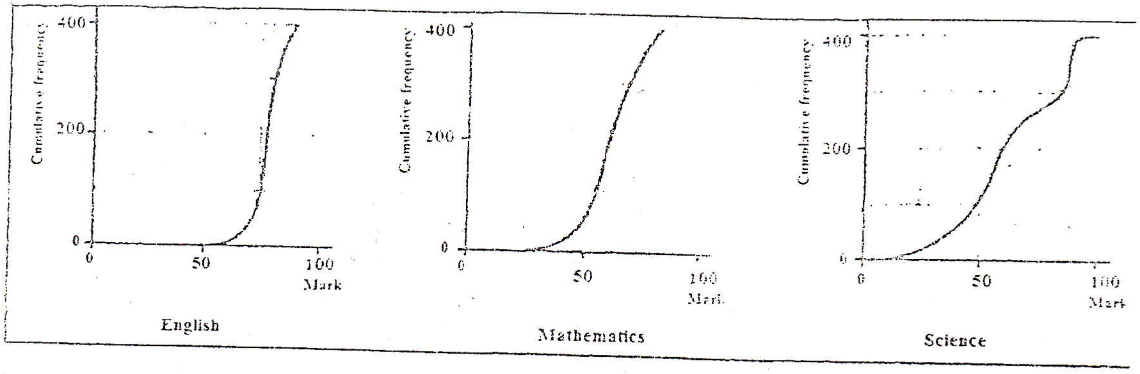
If $f: E \to F$

And dim E is finite then dim E = dim f + dim Im f

Thus dim E $\Re^3 = dim\ ker\ f + dim\ ker\ f \Leftrightarrow dim\ Im\ f = 3 - 1 = 2$

As dim f = dim $\Re^2$ $and\ Im\ f \subset \Re^2\ then\ Im\ f \Rightarrow \Re^2$

15. Examinations in English, Mathematics and Science were taken by 400 students. Each examination was marked out of 100 and the cumulative graphs illustrating the results are shown below. 6 marks



a) In which subject was the median mark the highest?

b) In which subject was the interquartile range of the marks the greatest?

c) In which subject did approximately 75% of students score 50 marks or more?

Answer:

a) The median (M) will occupy $\frac{400^{th}}{2} = 200^{th}$ $place,$

b) The first quartile (Q $_1$) the $\frac{400^{th}}{4} = 100^{th}$

The second quartile (Q_2) $\frac{400^{th}x3}{4} = 300^{th}$

Thus referring to the diagrams:

- ✓ The highest mediane is for English
- ✓ The greatest ecart interquartile is science

The subject where 75% of students obtained the marks equal to 50 and more is science.

N.B: Use the above diagrams in your answer booklet to show your working.

SECTION B: Attempt ANY THREE questions (45 marks)

16. a) Find the length of the curve $y = \frac{(e^x+e^{-x})}{2}$ from x = 0 to x = 1. 5 marks

b) Evaluate $\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n$ 2 marks

c) Evaluate the volume generated by revolving the region bounded by the graph of $y = e^x$ from x = 0 to x = 1 around x –axis. 8 marks

Answer:

a)

Length = $\int_0^1 \frac{(e^x+e^{-x})}{2}dx = \frac{1}{2}\int_0^1(e^x + e^{-x})dx = \frac{1}{2}[e^x - e^{-x}]_0^1 = \frac{1}{2}\left(e - \frac{1}{e}\right) = 1.165$

b) $\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n = \left(1+\frac{2}{\infty}\right)^\infty = 1^\infty I.F$

Let $y = \lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n$

$\ln y = \lim_{n\to\infty} \ln\left(1+\frac{2}{n}\right)^n \Leftrightarrow \ln y = \lim_{n\to\infty} n\ln\left(1+\frac{2}{n}\right) = \infty.0$

$\ln y = \lim_{n\to\infty} \frac{\ln\left(1+\frac{2}{n}\right)}{\frac{1}{n}}$

$\ln y = \lim_{n\to\infty} \frac{\ln\left(1+\frac{2}{n}\right)'}{\left(\frac{1}{n}\right)'} = \lim_{n\to\infty} \frac{\frac{-\frac{2}{x^2}}{\frac{x+2}{x}}}{} x - \frac{x^2}{1} = \lim_{n\to\infty} \frac{2x}{x+2} = \lim_{n\to\infty} \frac{2}{1+\frac{1}{x}} = 2$

$\ln y = 2$

$y = e^2$

17. a) In the group of 12 international referees there are three from Africa, four from Asia and five from Europe. To officiate at a tournament, three referees are chosen at random from the group. Calculate the probability that

i) a referee is chosen from each continent

ii) exactly two referees are chosen from Asia.

ii) the three referees are chosen from the same continent. 4.5 marks

b) For a given set of data (x, y) it is known that means $\overline{x} = 10$ *and* $\overline{y} = 4$. The gradient of the regression line y on x is 0.6. Find the equation of this regression line and estimate y when x = 12. **6 marks**

c) Sketch the graph of $x^2+y^2-2x+8y+13=0$. 4.5 marks

Answer:

a) n = 12

Africa = 3

Asia = 4

Europe = 5

$n(s) = {}^{12}C_3 = \frac{12!}{3!9!} = \frac{12x11x10}{3x2} = 220$

i) $n(E) = {}^3C_1 \times {}^4C_1 \times {}^5C_1 = 3x4x5 = 60$

$P = \frac{n(E)}{n(S)} = \frac{60}{220} = 0.2$

ii) Probability that 2 referees are from asia exactly

$n(E) = ({}^4C_2 x {}^3C_1) + ({}^4C_2 x {}^5C_1) = 18+30 = 48$

$P = \frac{48}{220} = 0.21$

$n(E) = {}^3C_3 + {}^4C_3 + {}^5C_3 = 1+4+10 = 15$

$P = \frac{15}{220}$

b) $\overline{x} = 10, \overline{y} = 4$

Gradient of regression line of y on x is on 6

$y-\overline{y} = a(x-\overline{x})$ y *on* x

y – 4 = 0.6 (x-10)

y -4 = 0.6x-6

y = 0.6x -2

y = 0.6x12 -2 = 5.2

c)

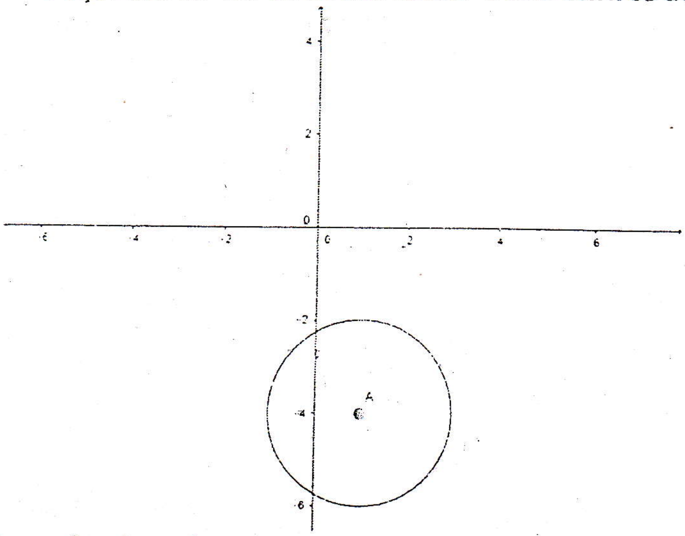
$x^2+y^2-2x+8y+13=0 \Leftrightarrow x^2-2x+y^2+8y=-13$

$\Leftrightarrow (x^2 - 2x) + (y^2 + 8y) = -13$

$\Leftrightarrow (x^2 - 2x + 1) - 1 + (y^2 + 8y + 16) - 16 = -13$

$\Leftrightarrow (x-1)^2 + (y+4)^2 = -13+1+16 \Leftrightarrow (x-1)^2+(y+4)^2 = 4$

The equation for the circle has radius 2 and centred at (1, -4)



18. a) Determine the real number a so that the following lines in the plane intersect x-y+1 = 0; 2x-y+2 = 0 and ax-y+3 =0. What is the intersection point? **3 marks**

b) Use the vector product to find the area of the triangle with vertices A(3, 0, -1), B(4, 2, 5) and C(7, -2, 4). **5 marks**

c) Prove that $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$.

d) Solve the equation: sin3x – cos2x = 0 for x∈ℜ **7 marks**

Answer:

a) We have to solve the equations

$$\begin{cases} x - y - 1 = 0 \\ 2x - y + 2 = 0 \\ ax - y + 3 = 0 \end{cases} \Rightarrow \begin{cases} x = -1 \\ y = 0 \\ ax - y + 3 = 0 \end{cases}$$

Thus a (-1) + 3 -0 ⇒ a = 3

The point of intersection has coordinates (-1, 0)

b) If A(3, 0, -1) B(4, 2, 5) and C(7, -2, 4)

Then the area of ΔABC is $\frac{1}{2}\|\overrightarrow{AB} \cap \overrightarrow{AC}\|$

$\overrightarrow{AB}$ =(1, 2, 6) and $\overrightarrow{AC} = (4, -2, 5)$

$$\|\overrightarrow{AB} \cap \overrightarrow{AC}\| = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 6 \\ 4 & -2 & 5 \end{vmatrix} = 22\vec{i} + 19\vec{i} - 10\vec{i}$$

Thus the area is $\frac{1}{2}\sqrt{22^2 + 19^2 + 10^2} = \frac{1}{2}\sqrt{945}$

c) $\cos\left(\frac{\pi}{2} - \theta\right) = \cos\frac{\pi}{2}\cos\theta + \sin\frac{\pi}{2}\sin\theta = \sin\theta$

d) Sin 3x-cos2x = 0 ⇒ sin 3x = cos2x

$\Rightarrow \cos\left(\frac{\pi}{2} - 3x\right) = \cos 2x \Rightarrow \frac{\pi}{2} - 3x = +2x + 2k\pi, k \in \Re$

$\frac{\pi}{2} - 3x = 2x + 2k\pi \Rightarrow x = \frac{\pi}{10} \pm 2k\pi$

$S = \left\{\frac{\pi}{10} + 2k\frac{\pi}{5}, \frac{\pi}{2} \pm 2k\pi\right\} k \in \Re$

19. a) Let $z = \frac{1+i\sqrt{3}}{1+i}$ be a complex number. Evaluate z^{2008} 8 marks

b) Let $\varphi: \Re^2 \to \Re^2: (x, y) \to \varphi(x, y) = (2x + 3y, x + 2y)$

i) Prove that φ is linear transformation of the vector space $(\Re, \Re^2, +)$.

ii) Show that φ is a bijection (one –to – one and onto) and define its inverse (φ^{-1})? 7 marks

Answer:

a) $\delta: \Re^2 \to \Re^2: (x, y) \to \delta(x, y) = (2x + 3y, x + 2y)$

δ *is linear if* $\forall$ *the real* α, β *and two vectors* $\vec{u}(x, y)$ *and* $\vec{v}(a, b)$ *of* $\Re^2$

We have $\delta(\alpha\vec{u} + \beta\vec{v}) = \alpha\delta\vec{u} + \beta\delta\vec{u} + \beta\delta\vec{v}$

$\delta(\alpha\vec{u} + \beta\vec{v}) = \delta(\alpha\vec{u} + \beta a, \alpha y + \beta b)$

$= [2(\alpha x + \beta a) + 3(\alpha y + \beta b), \alpha x + \beta a + 2(\alpha y + \beta b)]$

$= \alpha(2x + 3y, x + 2y) + \beta(2a + 3b, a + 2b) = \alpha\delta(x, y) + \beta\delta(a + b)$

$= \alpha\delta(\vec{u}) + \beta\delta(\vec{v}) \Leftrightarrow$ *linear*

Let show that δ is the bijection. Let be $(a, b) \in \Re^2$ *existance – t – one unique* $(x, y) \in \Re^2$ such that $\delta(x, y) = (a, b)$

$\delta(x, y) = (a, b) \Leftrightarrow (2x + 3y, x+2y) = (a, b)$

$\Leftrightarrow \begin{cases} 2x + 3y = a \\ x + 2y = b \end{cases}$

$\Leftrightarrow \begin{cases} x = 2a - 3b \\ y = -a + 2b \end{cases}$

Being given (a, b), *there exists a unique* $(x, y) = (2a + 3b, -a + 2b)$ *such that* $\delta(x, y) = (a, b)$

Thus δ *is bijective, from the previous, we deduce*

$\delta^{-1}: \Re^2 \to \Re^2 : (x, y) \to (2x-3y, -x+2y)$

N.B : To show that δ is bijection and defines δ^{-1} we can inverse the matrix $M = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ of δ and M^{-1} deduced δ^{-1}

20. Prove that the set of all complex numbers that have absolute value (modulus) 1 forms a commutative group with respect to multiplication. 15 marks

Answer:

Let note S the set of complex numbers of module 1 i.e

$S = \{Z \in \mathbb{C} : |z| = 1\}$

$S \neq \{\ \}$ *because* $z = 1 \in S$

To show that S is a commutative group. Let show that S is stable and closed for the multiplication: That this law is commutative symmetric

S is stable because $\forall Z \in S, \forall Z' \in S$ *we have*

1. The law is commutative and associative as it is in C
2. Let e be the neutral element of S for the multiplication

i.e $\forall Z \in S$ $\quad$ Z.e = z = e.z we have

$z.e = z \Leftrightarrow z.e - z = 0 \Leftrightarrow z(e-1) = 0 \Leftrightarrow e - 1 = 0$ because $z \neq 0, z \in s) \Leftrightarrow e = 1$

Thus 1 is the neutral element.

3. Verify if the law is symmetric

Let z = a+bi ∈ S

Let find Z' = x+yi ∈ S such that ZZ' = 1

ZZ' = 1 ⇔ (a+bi)(x+yi)

$\Leftrightarrow \begin{cases} ax - by = 1 \\ bx + ay = 0 \end{cases}$

$\Leftrightarrow \begin{cases} x = \frac{a}{a^2+b^2} \\ y = -\frac{b}{a^2+b^2} \end{cases}$

As $a^2 + b^2 = 1$ we have $\begin{cases} x = a \\ y = -b \end{cases}$

Thus the inverse of Z is its conjugate Z'

ADVENCED LEVEL MATHEMATICS NATIONAL EXAMINATION PAPER 2009

SECTION A: Attempt all questions. (55 marks)

01. a) Find A^{I} for $A = \begin{bmatrix} 3 & 2 \\ 4 & -3 \end{bmatrix}$

b) Let T:V⟶W be a linear transformation of real vector spaces. Find T(v) and T(w) if T(v+2w) = 3v-w and T(v - w)= 2v – 4w. 3.5 marks

Answer:

a) $A = \begin{bmatrix} 3 & 2 \\ 4 & -3 \end{bmatrix}$

$\Rightarrow \det A = \begin{vmatrix} 3 & 2 \\ 4 & -3 \end{vmatrix} = -9+8 = -1 \neq 0 \Rightarrow A^{-1} = \frac{-1}{1}\begin{pmatrix} -3 & -2 \\ 4 & 3 \end{pmatrix} = -\begin{pmatrix} -3 & -2 \\ 4 & 3 \end{pmatrix} =$

$\begin{pmatrix} 3 & 2 \\ -4 & -3 \end{pmatrix}$

b) $\begin{cases} T(v+2w) = 3v - w \\ T(v-w) = 2v - 4w \end{cases} \Leftrightarrow \begin{cases} T(v) + 2T(w) = (3v - w) \\ T(v) - T(w) = (2v - 4w) \end{cases} \Leftrightarrow \begin{cases} T(v) = \frac{7}{3}v - 3w \\ T(w) = \frac{1}{3}v + w \end{cases}$

02. a) Find the derivative by the limit process: $f(x) = x^3 - x$

b) Show that this function is continuous $f(x) = |x+2| - 5$ 6 marks

Answer:

a) $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{f(x+h)^3 - f(x+h) - x^3 + x}{h} = \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h} = 3x^2 - 1$

b) $f(x) = |x+2| - 5 = \begin{cases} x + 2 - 5; if\ x > -2 \\ -5; if\ x = -2 \\ -x - 2 - 5; if\ x < -2 \end{cases} = \begin{cases} x - 3; x > -2 \\ -5; x = -2 \\ -x - 7; x < -2 \end{cases}$

Since f discontinuous on $]-2, +\infty[\cup]-\infty, -2[$, as polynomial function. Let study the continuity at the point x =-2, $\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} (-x - 7) = -5$

$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} (x - 3) = = 5$

as the limit at the left and the limit at the right at the point $x = -2$

Conclusion: f is continuous on ℜ

03. Let $f(x) = \frac{\sqrt{x+c^2}}{x} - \frac{c}{x}$ where c > 0. What is the function of f? How can you define f at x =0 in order for f to be continuous there? 3 marks

Answer:

Define f at x = 0 in order for f to be continuous there may calculate $\lim_{x\to 0} f(x)$

$\lim_{x\to 0} \frac{\sqrt{x-c^2}}{x} - \frac{c}{x} = \frac{0}{0}$ I.F

$\lim_{x\to 0} \frac{\sqrt{x+c^2}-c}{x} = \lim_{x\to 0} \frac{\sqrt{x+c^2}-c}{x} = \lim_{x\to 0} \frac{(\sqrt{x+c^2}-c)(\sqrt{x+c^2}+c)}{x\,(\sqrt{x+c^2}+c)} = \lim_{x\to 0} \frac{x+c^2-c^2}{x\,(\sqrt{x+c^2}+c)} = \lim_{x\to 0} \frac{1}{(\sqrt{x+c^2}+c)} =$

$\frac{1}{2c}$

$$f(x) = \begin{cases} \frac{\sqrt{x-c^2}}{x} - \frac{c}{x} & if\ x \neq 0 \\ \frac{1}{2c} & if\ x = 0 \end{cases}$$

04. A group of 50 people was asked which of the three newspapers: ''Imvaho Nshya'', ''Umuseso'' or ''NewTimes'' they read. The results showed that 25 read Imvaho Nshya, 16 read Umuseso, 14 read NewTimes, 5 read both Imvaho Nshya and Umuseso, 4 read both Umuseso and NewTimes, 6 read both Imvaho Nshya and NewTimes, and 2 read all three papers.

a) Represent these data on a Venn diagram.

b) Find the probability that a person selected at random from this group reads:

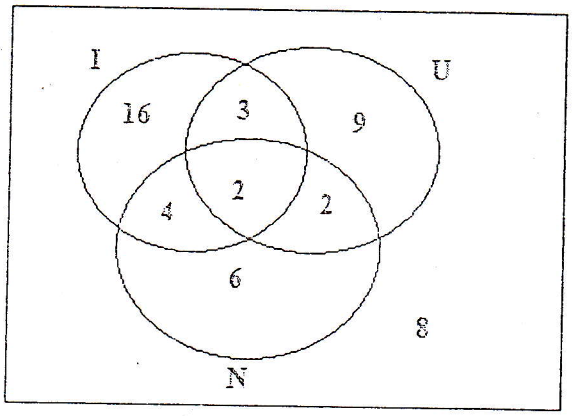
i) At least one of the three newspapers

ii) Only one of the newspapers

iii) Only Imvaho Nshya. 5 marks

Answer:

a) **I be the set of people who read "imvaho nshya"**
U be the set of people who read '' Newtimes''



b) **i) p (a person reads at least one of the three news papers) = 1-p (person does not read these news papers) =** $1-\frac{8}{50} = \frac{42}{50} = \frac{21}{25}$

ii) p (person reads-only one of the newspapers) = $\frac{16}{50} + \frac{9}{50} + \frac{6}{50} = \frac{31}{50}$

iii) p(person reads only Imvaho nshya) = $\frac{16}{50} = \frac{8}{25}$

05. Locate any relative extremum and inflection point of the real function $f(x) = \frac{x^2}{2} - \ln x$. 3 marks

Answer:

$f(x) = \frac{x^2}{2} - \ln x$ Domf =]0, ∞[

$f'(x) = x - \frac{1}{x}$ and $f''(x) = 1 + \frac{1}{x^2} = \frac{x^2+1}{x^2}$

$f'(x) = 0 \Leftrightarrow x = \pm 1$; $f''(x) \neq 0\ \forall x \in Domf$

x	0			1		
f'(x)		- - - - -		0 + + + +		
f''(x)	+	+ +		+	+	+
f(x)	+∞			½		

f admits a minimum at point x = 1. That minimum value is ½ (absolute) :
$f(1) = 1 - \frac{1}{2} - \ln 1 = \frac{1}{2}$.

The function has no point of inflection because $f''(x) \neq 0\ \forall x \in Dom\, f$

06. In Euclidian space $(\Re^3, +, .)$:

a) Find the vector, symmetric and parametric equations of the line passing through P (1, 0, -3) and parallel to the line with the parametric equations $\begin{cases} x = -1 + 2t \\ y = 2 - t \\ z = 3 + 3t \end{cases}$

b) Find the z coordinate of B if the distance between A(4, 1, -2) and B(1, -1, z) is $\sqrt{17}$.

4.5 marks

Answer:

a) The director vector of the asked straight line is $\vec{u}(2, -1, 3)$

Let note the straight line D

The vector equation is

$\vec{Px} = \lambda\vec{u}$ or $\vec{0x} = \lambda\vec{u} + \vec{oP}$ with x any point of that line

The parametric equation $\begin{cases} x = 2\lambda + 1 \\ y = -\lambda \\ z = 3\lambda - 3 \end{cases}$

The system of Cartesian equation is $\frac{x-1}{2} = -y = \frac{z+3}{3}$

b) $d(A, B) = \|\vec{AB}\| = \sqrt{(4-1)^2 + (1+1)^2 + (-2-z)^2} = \sqrt{9 + 4 + (2+z)^2} = \sqrt{17 + 4z + z^2} = d(AB) = \sqrt{17}$

$17 + 4z + z^2 = 17 \Leftrightarrow z(4+z) = 0 \Leftrightarrow z = -4$ or $z = 0$

Thus $B(1, -1, 0)$ *or* $B(1, -1, -4)$

07. The letters of the word MATHEMATICS are written, one on earth of 11 separate cards. The cards are laid out in a line.

a) Calculate the number of different arrangements of these letters.

b) Determine the probability that the vowels are placed together. 3 marks

Answer:

a) $\{M, A, T, H, E, M, A, T, I, C, S\}$

The number of possible arrangement is $\frac{11!}{2!2!2!} = \frac{11!}{8}$

b) **If the vowels AEAI must follow one another, we have the bloc AEAI as "one letter" thus the number of arrangement =** $\frac{8!}{2!2!}$ **but as the vowels can switch** between them $\frac{4!}{2!}$ ways the number of possible arrangement is finally $\frac{4!}{2!}x\frac{8!}{2!2!} = \frac{4!8!}{2!2!2!}$

$\Rightarrow P(\text{vowels together}) = \frac{4!8!2!2!2}{2!2!2!11!} = \frac{4}{165}$

08. In Euclidian space $(\Re^3, +, .)$ find all points C on the line through A (1, 1, -20) and B(2, 0, 1) such that $\|\overrightarrow{AC}\| = 2\|\overrightarrow{BC}\|$ 3.5 marks

Answer:

$\|\overrightarrow{AC}\| = 2\|\overrightarrow{BC}\| \Leftrightarrow (x-1)^2+(y-1)^2y+(z+2)^2 = 4[(x-2)^2+y^2+(z-1)^2]$ with C(x, y, z)

As C lies on straight line AB then

$$\begin{cases} x = 1 + t \\ y = 1 - t \quad \text{For } t \in \Re \\ z = -2 + 3t \end{cases}$$

We obtain the system

$$\begin{cases} (x-1)^2 + (y-1)^2 + (z+2)^2 = 4[(x-2)^2 + y^2 + (z-1)^2] \\ x = 1 + t \\ y = 1 - t \\ z = -2 + 3t \end{cases}$$

By replacing x, y, z by their values with respect to t we obtain the equation of second degree:

$3t^2-8t+4 = 0$ in t. After solving we have $t = 2$ or $t = \frac{2}{3}$

Thus C(3. -1, 4) or C $(\frac{5}{3}, \frac{1}{3}, 0)$

Or Method

$$A\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, B\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

Let c(x, y, z)

$\|\overrightarrow{AC}\| = 2\|\overrightarrow{BC}\| \Leftrightarrow \overrightarrow{AC} = \pm\overrightarrow{BC}$

$\overrightarrow{AC} = 2\overrightarrow{BC}$

Case 1: $\Leftrightarrow \begin{pmatrix} x-1 \\ y-1 \\ z+2 \end{pmatrix} = 2\begin{pmatrix} x-2 \\ y \\ z-1 \end{pmatrix} \Leftrightarrow \begin{cases} x - 1 = 2x - 4 \\ y - 1 = 2y \\ z + 2 = 2z - 2 \end{cases}$

x=3, y = -1, z = 4

Thus C(3, -1, 4)

Case 2:

$\overrightarrow{AC} = -2\overrightarrow{BC}$

$$\begin{pmatrix} x-1 \\ y-1 \\ z+2 \end{pmatrix} = -2\begin{pmatrix} x-2 \\ y \\ z-1 \end{pmatrix} \Leftrightarrow \begin{cases} x - 1 = -2x - 4 \\ y - 1 = -2y \\ z + 2 = -2z + 2 \end{cases}$$

$\Leftrightarrow \frac{5}{3}, y = \frac{1}{3}, z = 0$

Thus $C\left(\frac{5}{3}, \frac{1}{3}, 0\right)$

09. A manufacturer wants to design an open box having h meters as height, a square base with side x meters and surface area of 108 square meters. What dimensions will produce a box with maximum volume? **3.5 marks**

Answer:

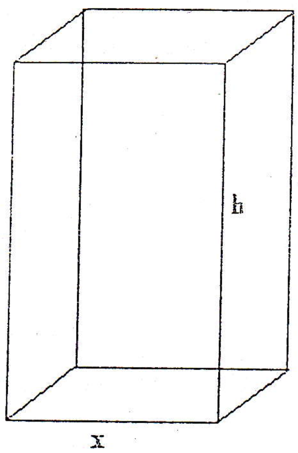
We know that $V = x^2.h$

The surface area is $S = 4hx+x^2 = 108$

Thus $h = \frac{27}{x} - \frac{1}{4}x$

$V = x^2[\frac{27}{x} - \frac{1}{4}x] = 27x-\frac{1}{4}x^3$

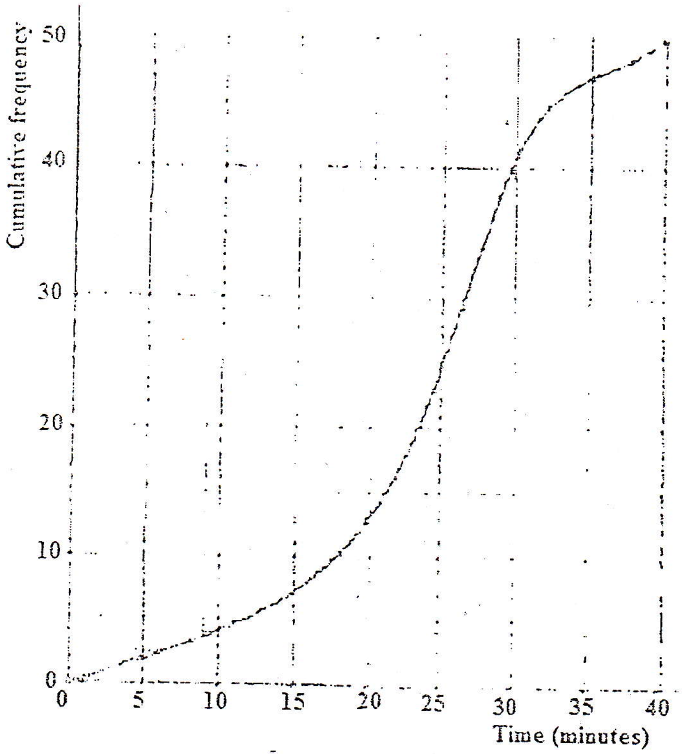
$V'(x) = 27 - \frac{3}{4}x^2$



x	0		6	
V'(x)		+ + +	0	- - - - -
V(x)	0	↗	108	↘

The volume is maximum for x = 6m and h = 3m and the volume V = 6 x 6 x 3 = $108m^3$

10. The cumulative frequency curve has been drawn from information about the amount of time in minutes spent by 50 people in a supermarket on a particular day.



a) Construct the cumulative frequency table taking boundaries $\leq 5, \leq 10, \leq 15, \ldots$..
b) How many people spent between 17 and 27 minutes in the supermarket?
c) 60% of the people spent less or equal to t minutes. Find t.
d) 60% of the people spent longer than s minutes. Find s.
e) Estimate the median. 4.5 marks

Answer:

a) The table of cumulative frequencies, we note $]a, b[$ by $a - b$

Time(minutes)	0-5	5-10	10-15	15-20	20-25	25-30	30-35	35-40
Cumulative frequencies	2	4	7	13	25	41	47	50

b) On the diagram, we observe that 31 people spent under 27 minutes otherwise 9 people spent under 17 minutes. Thus 30-9 = 21 spent between 17 and 27 minutes.

c) 60% of 50 is $\frac{60x50}{100} = 30$ thus 30 people spent t = 26 minutes in the super market at more.

d) 60% of people
$\Rightarrow$ 30 and there are 30 people who spent longer than S = 23 minutes in the super market (see graph)

e) The median is 25 because, using the horizontal line
$y = \frac{n}{2} (where\ n = 50)$

$y = \frac{50}{2} = 25$, *this line meets the graph (the curve) at* $t = 25$, *that is, the median is* 25

11. a) Find the roots of the complex number $z = \sqrt{2} + i\sqrt{2}$. Write answers in polar form.
b) If $T: \Re^2 \to \Re^2$ transforms each point to its reflection in the x- axis, prove that T is linear and write its matrix in the standard basis of the vector space $(\Re, \Re^2, +)$. **3 marks**

Answer:

a) $z = \sqrt{2} + i\sqrt{2}$ $\quad |z| = \sqrt{(\sqrt{2})^2 + (\sqrt{2})^2} = 2$

$$\begin{cases} \cos\theta = \frac{\sqrt{2}}{2} \\ \sin\theta = \frac{\sqrt{2}}{2} \end{cases} \Rightarrow \theta = \frac{\pi}{2} + 2k\pi, \forall k \in Z$$

The roots of **Z are** of the form

$Z_k = \sqrt{2}\left[\cos\left(\frac{\pi}{8} + k\pi\right) + i\sin\left(\frac{\pi}{8} + k\pi\right)\right] \qquad k = 0, 1$

i.e $Z_0 = \sqrt{2}\left[\cos\left(\frac{\pi}{8}\right) + i\sin\left(\frac{\pi}{8}\right)\right]$

$Z_1 = \sqrt{2}\left[\cos\left(\frac{9\pi}{8}\right) + i\sin\left(\frac{9\pi}{8}\right)\right]$

b) $T: \Re^2 \to \Re^2: (x, y) \to T(x, y) = (x, -y)$

T is linear if $T[(x, y) + (x', y')] = T(x, y) + T(x', y')$

and $T(\alpha(x, y)) = \alpha T(x, y)$ *for* $\alpha \in \Re\ (x, y)\ \textcircled{e} x', y' \in \Re^2)$

Or $T((x,y)+(x', y')) = T(x+x', y+y') = (x+x',y+y') = (x+x', -y-y') = (x, -y)+(x', -y')$
$= T(x, y) + T(x', y')$

$T(\alpha(x, y)) = T(\alpha x, \alpha y) = (\alpha x, -(\alpha y)) = \alpha(x, -y) = \alpha T(x + y)$

Conclusion : T is linear

The canonic base of $\Re^2$ *is* $\beta((1, 0); (0, 1))$

We have $T(1, 0) = (1, 0) = 1(1,0)+0(0,1)$

$T(0,1) = (0, -1) = 0(1, 0)-1(0,1)$

The matrix of T is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Other method

$z = \sqrt{2} + i\sqrt{2}$

Let $t = x + iy$ such that $t^2 = Z$

$(x+yi)^2 = \sqrt{2} + i\sqrt{2}$

$x^2 - y^2 + 2xyi = \sqrt{2} + i\sqrt{2}$

Thus $x^2 - y^2 = \sqrt{2}$ (1)

$2xy = \sqrt{2}$ (2)

In addition $|Z^2| = |Z| \Rightarrow x^2 + y^2 = 2$ (3)

We obtain the system $\begin{cases} x^2 - y^2 = \sqrt{2} \\ 2xy = \sqrt{2} \\ x^2 + y^2 = 2 \end{cases}$

From (1)+(3) we find: $x = \pm\sqrt{\frac{2+\sqrt{2}}{2}}$

From (3)-(1) we find: $y = \pm\sqrt{\frac{2-\sqrt{2}}{2}}$

As xy> $0, x$ *and* y *are of the same sign then we have*

$t_1 = \sqrt{\frac{2+\sqrt{2}}{2}} + \sqrt{\frac{2-\sqrt{2}}{2}}i$

$t_2 = -\sqrt{\frac{2+\sqrt{2}}{2}} - \sqrt{\frac{2-\sqrt{2}}{2}}i$

12. a) Evaluate the integrals i) $\int_{-1}^{2} 2^x dx$ ii) $\int_3^6 \frac{x}{\sqrt{x^8-8}} dx$

b) Find the values of the constants a and b such that $\lim_{x\to 0} \frac{\sqrt{a+bx}-\sqrt{3}}{x} = \sqrt{3}$ 3 marks

Answer:

a) i) $\int_{-1}^{2} 2^x dx = \left[\frac{1}{ln2} 2^x\right]_{-1}^{2} = \frac{1}{ln2}(2^2 - 2^{-1}) = \frac{7}{2ln2} = \frac{7}{ln4}$

ii) $\int_3^6 \frac{x}{\sqrt{x^8-8}} dx = \left[\sqrt{x^2-8}\right]_3^6 = \sqrt{36-8} - \sqrt{9-8} = \sqrt{28} - 1$

b) $\lim_{x\to 0} \frac{\sqrt{a+bx}-\sqrt{3}}{x} = \sqrt{3}$ this can happen if we have the I.F $\frac{0}{0}$ *i.e* $a = 3$

$\lim_{x\to 0} \frac{\sqrt{a+bx}-\sqrt{3}}{x} = \lim_{x\to 0} \frac{3+bx-3}{x(\sqrt{3+bx}+\sqrt{3}} = \frac{b}{2\sqrt{3}}$

Thus $\frac{b}{2\sqrt{3}} = \sqrt{3}$ this gives b = 6

Conclusion $\lim_{x\to 0} \frac{\sqrt{a+bx}-\sqrt{3}}{x} = \sqrt{3}$ if a = 3 and b = 6

13. Find x and y so that $\begin{bmatrix} 1 & 3 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} x & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} y & 7 \\ y & -6 \end{bmatrix}$ 1.5 marks

Answer:

$\begin{pmatrix} 1 & 3 \\ -2 & -2 \end{pmatrix} x \begin{pmatrix} x & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} y & 7 \\ y & -6 \end{pmatrix}$

$\Leftrightarrow \begin{pmatrix} x+9 & 1+6 \\ -2x-6 & -2-4 \end{pmatrix} = \begin{pmatrix} y & 7 \\ y & -6 \end{pmatrix}$

$\Leftrightarrow \begin{cases} x+9 = y \\ -2x-6 = y \end{cases} \Leftrightarrow \begin{cases} x = -5 \\ y = 4 \end{cases}$

14. Using matrix inverse solve the system for real numbers x, y and z:

$\begin{cases} x + 3y - 2z = 1 \\ y + 5z = 2 \\ -2x - 6y + 7z = 0 \end{cases}$ 3 marks

Answer:

$\begin{cases} x + 3y - 2z = 1 \\ y + 5z = 2 \\ -2x - 6y + 7z = 0 \end{cases} \Leftrightarrow \begin{pmatrix} -1 & 3 & -2 \\ 0 & 1 & 5 \\ -2 & -6 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

The matrix of the system is $A = \begin{pmatrix} -1 & 3 & -2 \\ 0 & 1 & 5 \\ -2 & -6 & 7 \end{pmatrix}$, the determinant $A = 3 \neq 0$

$CofA = \begin{pmatrix} 37 & -10 & 2 \\ -9 & 3 & 0 \\ 17 & -5 & 1 \end{pmatrix}$

$AdjA = (Cof(A))' = \begin{pmatrix} 37 & -9 & 17 \\ -10 & 3 & -5 \\ 2 & 0 & 1 \end{pmatrix}$ the solution of the system is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 37 & -9 & 17 \\ -10 & 3 & -5 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 37-18 \\ -10+6 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{19}{3} \\ -\frac{4}{3} \\ \frac{2}{3} \end{pmatrix}$$

$Therefore\ x = \frac{19}{3}, y = -\frac{4}{3}, z = \frac{2}{3}$

15. Let the binary operation *be defined on Z (ring of integers) by x*y = x+xy+y
   a) Calculate 2*(-1), (-1)*9, and 6*1
   b) Determine whether *is commutative, associative or neither.
   c) Determine whether or not there exist an identity for *.5 marks

Answer:

a) x*y = x+xy+y
   2*(-1) = 2+2(-1)+(-1) = 2-2-1 = -1
   (-1)*9 = -1+(-1)9+9 = -1-9+9 = -1
   6*1 = 6+6(1) + 1 =6+6+1 = 13

b) * is commutative if $\forall x, y \in Z, x * y = y * x$
   x*y = x+xy+y = y+yx+x (+ is commutative)
   = y+yx+x (is commutative)
   = y*x(definition of *)
   Conclusion: * is commutative n Z
   * is associative if x*(y*z) = (x*y)*z, $\forall x, y, z \in Z$
   x*(y*z) = x+x(y*z)+y*z
   = x+x(y+yz+z) +y+yz+z
   = x+xy+yzx+xz+y+yz+z
   = x+xy+y+(xy+x+y)z + z
   = x*y+(x*y)z+z
   = (x*y)*z
   Conclusion : * is associative in Z

c) Let e the neutral element. $\forall x \in Z$ $we\ must\ have\ x * e = x = e * x$
   $as * is\ commutative, it\ is\ enough\ that\ x * e = x$
   Therefore x*e = x ⇔x+xe+e x
   ⇔ xe+e = 0
   ⇔(x+1)e = 0
   ⇔ x+1 $\neq 0 \Rightarrow e = 0\ provided\ x \neq 1\ therefore\ e = 0$ is the identity element for the operation *

SECTION B: ANSWER ANY THREE QUESTIONS (45 marks)

16. a) A population (P) of bacteria is changing at a rate of $\frac{dp}{dt} = \frac{3000}{1+0.25t}$ where t is the time in days. The initial population when t = 0 is 1000. Write an equation that gives you the population at any time t and find the population when t =3 days. 5 marks

b) Find equations of the tangent lines to the graph of $f(x) = \frac{x+1}{x-1}$ that are parallel to the line $2y+x = 6$. Find critical numbers as well as asymptotes of the function. 7 marks

c) Find the equation of circle with center (2, 2) whose graph passes through the point (3, 5). 3 marks

Answer:

a) $\frac{dp}{dt} = \frac{3000}{1+0.25t} \Rightarrow \int dp = \int \frac{3000}{1+0.25t} dt = = 3000 \int \frac{dt}{\frac{4+t}{4}} = 3000 \int \frac{4dt}{4+t} = 12000 \int \frac{dt}{4+t}$

$= 12000 \ln|4+t| + C$

$dp = 12000 \ln|4+t| + C$ i.e $P(t) = 12000 \ln|4+0| + C = 1000$

$\Rightarrow 12000\ln 4 + C = 1000 \Rightarrow C = 1000 - 12000\ln 4$

$\Rightarrow C = -15635.5$ hence $P(t) = 12000 \ln|4+t| - 15635.5$

When $t = 3 \Rightarrow P(t) = 12000 \ln 7 = 1000$

Or

at $t = 0$, $P = 1000$

$\Rightarrow 1000 = 12000\ln 4 + C$

$C = 1000 - 12000\ln 4 = -15635.5$

Eq: $P(t) = 12000 \ln|4+t| - 15635.5$

At $t = 3$, $P(3) = 12000\ln 7 - 15635.5 = 7715$

b) The slope of the line $y = -\frac{1}{2}x + 3$ is equal to $-\frac{1}{2}$ thus the tangent on curve has a slope $f'(x_o) = -\frac{1}{2}$ therefore $f'(x) = \frac{x-1-x-1}{(x-1)^2} = \frac{-2}{(x-1)^2}$

$f'(x_0) = \frac{-2}{(x_o-1)^2} = \frac{-1}{2} \Leftrightarrow (x_o-1)^2 = 4 \Leftrightarrow x_o = 3$ or $x_o = -1$

thus we have two tangents of equations:

$T_1 \equiv y - f(3) = -\frac{1}{2}(x-3)$ or $T_1 \equiv y - 2 = -\frac{1}{2}(x-3)$

$T_2 \equiv y - f(-1) = -\frac{1}{2}(x+1)$ or $T_2 \equiv y = -\frac{1}{2}(x+1)$

$\left.\begin{matrix} f'(x) \neq 0 \\ f''(x) \neq 0 \end{matrix}\right\} \forall x \in Dom f \Rightarrow$ *no critical point.*

$A.H \equiv y = 1$ *because* $\lim_{x\to\infty} \frac{x+1}{x-1} = 1$

$V.A \equiv x = 1,$ *because* $\lim_{x\to 1} \frac{x+1}{x-1} = \infty$

No O.A

$f''(x) = \frac{4}{(x-1)^3}$

c) The equation of cercle is $(x-2)^2+(y-2)^2 = R^2$

As (3, 5) ∈ to the cercle, we have $(3-2)^2+(-5-2)^2 = R^2$

Let be $R^2 = 50$, thus $(x-2)^2+(y-2)^2$ 50 or $x^2+y^2-4y-42 = 50$

17. a) Consider the point P (3, 4) and the circle $x^2+y^2 = 25$.

i) Is P a point of the circle?

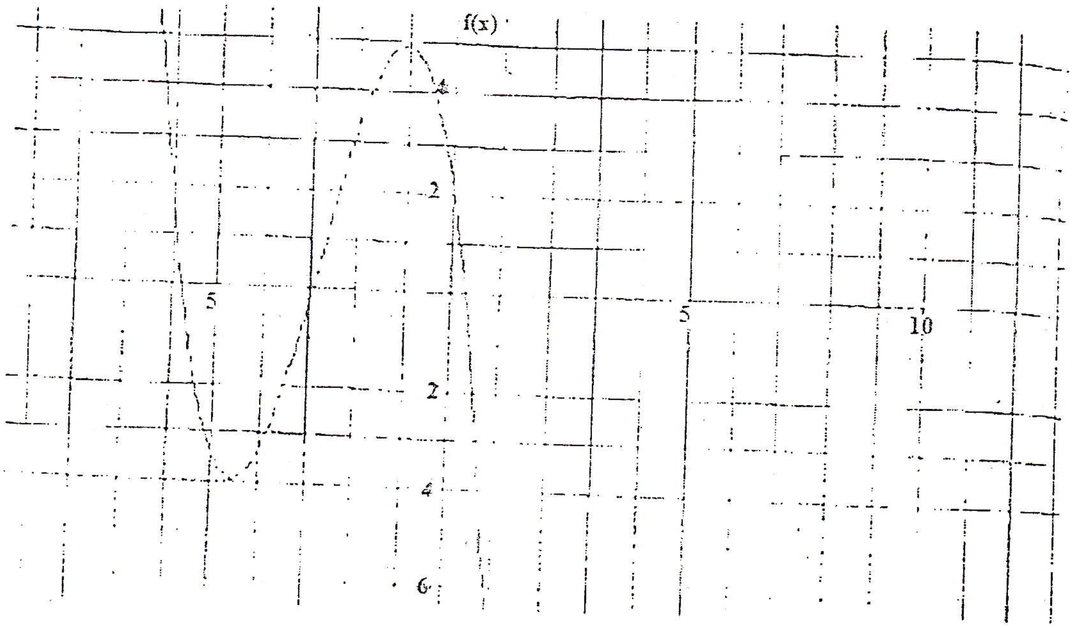
ii) What is the slope of the line joining P and O(0, 0)?

iii) Find an equation of the tangent line to circle at P.

iv) Let Q(x, y) be another point on the circle in the first quadrant. Find the slope m_x of the line joining P and Q in terms of x.

v) Calculate $\lim_{x\to o} m_x$. How does this number relate to your answer in part (ii). 7.5 marks

b) Study the following graph of a function f and answer the questions that follow:



i) Find the domain and the range (image) of the function f

ii) Find $\lim_{x\to\infty} f(x)$

iii) Determine whether the function is an application, one to one (injection), onto (surjection), bijection or neither. Justify your answer.

iv) Find intervals of increase and decrease for the function.

v) Find the derivative of f at x = -1. Provide justification to your answer.

vi) Can you guess the number of zeros of the function f? Give an interval (a, b) such that $|b-a| = 1$ in which each zero lies. **6.5 marks**

Answer:

a) f≡ $x^2 + y^2 = 25$

i) p(3,4) is a point of the cercle because $3^2+4^2 = 25$

ii) The slope of OP is $\frac{4}{3}$

iii) $T_p \equiv x_o x + y_o y = 25$ for $P(x_0, y_o) \in$ cercle thus $T_p \equiv 3x + 4y = 25$ *or* $y = -\frac{3}{4}x + \frac{25}{4}$

iv) Let Q(x, y) with $(x, y) \in \Re^2 x \Re^2$ *and* $x^2 + y^2 = 25$

$m_x = \frac{y-4}{x-3}$ *as* $Q \in$ *cercle we have* $y^2 = 25 - x^2 \Rightarrow y = \sqrt{25-x^2} (y > 0)$

Thus $m_x = \frac{\sqrt{21-x^2}-4}{x-3}$

v) $\lim_{x\to 3} m_x = \lim_{x\to 3} \frac{\sqrt{25-x^2}}{x-3} = \lim_{x\to 3} \frac{9-x^2}{x-3(\sqrt{25-x^2}+4)} = \lim_{x\to 3} \frac{-(3+x)}{(\sqrt{25-x^2}+4)} = -\frac{6}{8} = -\frac{3}{4}$

We see that the limit and the slope found in (ii) have the product equal to -1 thus OP is perpendicular to the tangent at point P

b) i) Dom f = $\Re$ *and* $Imf = \Re$

ii) $\lim_{x\to-\infty} f(x) = +\infty$ and $\lim_{x\to+\infty} f(x) = -\infty$

iii) f is an application because Im f = $\Re$ is not injective because two distinct points have the same image. Let use the horizontal line y =2 for example: It cuts the straight line in more than two points. f is surjection because Im j = $\Re$ *f is not bijective because n*
o injective.

iv) f increases on [-4, 5, -1] and decreases on $\left]-\infty,-\frac{9}{2}\right] \cup [1,+\infty[$

v) f has three zeros because the curve cut OX in three distinct points, those zeros belong to the intervals: $]-6,-5[\,]-3,-2[$ *and* $[0,1[$

18. a) Solve the following equations in the field of real numbers

i) ln(lnx) = 1

ii) ln(x+1) = ln(3x+1)-lnx

iii) $\log_2(x-1) = 5$

b) How many years, to the nearest year, will it take money to quadruple if it is invested at 20% compounded annually? 3.5 marks

c) Determine the point (s) in the interval $(0, 2\pi)$ at which the graph of the function f(x) = 2cosx + sin2x has horizontal tangent. 6.5 marks

Answer:

a) i) ln(lnx) = 1 and lnx = e $\Rightarrow$ $x = e^e$, $S = \{e^e\}$

ii) $\ln(x+1) = \ln(3x+1) - \ln x \Leftrightarrow \begin{cases} x > 0 \\ x > -1 \\ x > -\frac{1}{3} \\ \ln(x+1) = \ln\frac{(3x+1)}{x} \end{cases}$

$\Leftrightarrow \begin{cases} x > 0 \\ x + 1 = \frac{3x+1}{x} \end{cases} \Leftrightarrow x = 1-\sqrt{2} \Leftrightarrow S = \{1-\sqrt{2}\}$

iii) $\log_2(x-1) = 5 \Leftrightarrow \begin{cases} x > 1 \\ x - 1 = 2^5 \end{cases} \Leftrightarrow \begin{cases} x > 1 \\ x = 2^5 + 1 \end{cases} \Rightarrow x = 2^5 + 1 = 33$

Thus S = {33}

b) Let C_o be the initial capital placed at the rate of r% =

After the first year the capital becomes $C_1 = C_o + \frac{C_o x r}{100} = C_o\left(1 + \frac{r}{100}\right)$

After the second year it becomes $C_2 = C_1 + \frac{C_1 x r}{100} = C_o\left(\left(1+\frac{r}{100}\right)\right)^2$ the same reasoning conduct us to the formula: $C_n = \left(1 + \frac{r}{100}\right)^n$ of the capital after n years

If r = 20% and $C_n = 4C_o$, We have $4C_o = C_o\left(1 + \frac{20}{100}\right)^n \Leftrightarrow 4 = \left(1 + \frac{20}{100}\right)^n$

$\Leftrightarrow \ln 4 = n\ln(1+0.2) \Leftrightarrow n = \frac{\ln 4}{\ln 1.2} \approx 8$ *years*

c) The graph of f admits the horizontal tangent at $[(x_o, f(x))]$ if $f'(x_0) = 0$

$f'(x) = -2\sin x + 2\cos 2x$

$f'(x) = 0 \Leftrightarrow -2\sin x + 2(\cos^2 x - \sin^2 x) = 0$

$\Leftrightarrow -2\sin x + 2(1 - 2\sin^2 x) = 0$

$\Leftrightarrow -4\sin^2 x - 2\sin x + 2 = 0$

ii) $\lim_{x\to-\infty} f(x) = +\infty$ and $\lim_{x\to+\infty} f(x) = -\infty$

iii) f is an application because Im f = $\Re$ is not injective because two distinct points have the same image. Let use the horizontal line y =2 for example: It cuts the straight line in more than two points. f is surjection because Im j = $\Re$ *f is not bijective because n*
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Answer:

a) i) ln(lnx) = 1 and lnx = e $\Rightarrow x = e^e$, $S = \{e^e\}$

ii) $\ln(x+1) = \ln(3x+1) - \ln x \Leftrightarrow \begin{cases} x > 0 \\ x > -1 \\ x > -\frac{1}{3} \\ \ln(x+1) = \ln\frac{(3x+1)}{x} \end{cases}$

$\Leftrightarrow \begin{cases} x > 0 \\ x+1 = \frac{3x+1}{x} \end{cases} \Leftrightarrow x = 1-\sqrt{2} \Leftrightarrow S = \{1-\sqrt{2}\}$

iii) $\log_2(x-1) = 5 \Leftrightarrow \begin{cases} x > 1 \\ x-1 = 2^5 \end{cases} \Leftrightarrow \begin{cases} x > 1 \\ x = 2^5+1 \end{cases} \Rightarrow x = 2^5 + 1 = 33$

Thus S = {33}

b) Let C_o be the initial capital placed at the rate of r% =

After the first year the capital becomes $C_1 = C_o + \frac{C_o x r}{100} = C_o\left(1+\frac{r}{100}\right)$

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If r = 20% and $C_n = 4C_o$, We have $4C_o = C_o\left(1+\frac{20}{100}\right)^n \Leftrightarrow 4 = \left(1+\frac{20}{100}\right)^n$

$\Leftrightarrow \ln 4 = n\ln(1+0.2) \Leftrightarrow n = \frac{\ln 4}{\ln 1.2} \approx 8$ *years*

c) The graph of f admits the horizontal tangent at $[(x_o, f(x))]$ if $f'(x_0) = 0$

f'(x) = -2sinx+2cos2x

$f'(x) = 0 \Leftrightarrow -2\sin x + 2(\cos^2 x - \sin^2 x) = 0$

$\Leftrightarrow -2\sin x + 2(1-2\sin^2 x) = 0$

$\Leftrightarrow -4\sin^2 x - 2\sin x + 2 = 0$

$\Leftrightarrow$ -4[sinx+1][sinx-$\frac{1}{2}$] = 0

Thus $f'(x_o) = 0 \Leftrightarrow$ sin x = -1 or sin x = +$\frac{1}{2}$

x	0	π/6	5π/6	3 π/6	2 π
f'(x)	+++ 0	- - - - - 0	+ + + + 0	+++++	

The graph of f admits horizontal tangents at x = π/6 and x = 5 π/6

19. a) Express the complex numbers 3^{π} in the standard for a+bi 2 marks

b) Find all (real or complex) numbers x such that x^3=-8. 4 marks

c) Write an equation of the plane passing through P (1, 0, -1) with normal vector $\begin{pmatrix}2\\2\\-1\end{pmatrix}$ 4 marks

d) Discuss the domain of the function f(x) = $\sqrt{x^2-4mx+5m-1}$ in real variable x . 5 marks

Answer:

a) Z = $3e^{\pi i}$= 3(cosπ+isinπ) = 3(-1+0i) = -3

b) x^3 = -8 $\Leftrightarrow$ x = $(-8)^{\frac{1}{3}}$ we need to determine all cubic roots of -8 therefore -8 = $8e^{\pi i}$ = 8(cosπ+isinπ) = 8[cos(π+2kπ) + isin(π+2kπ)] and $\sqrt[3]{-8} = \sqrt[3]{-8}\left[cos\left(\frac{\pi+2k\pi}{3}\right) + isin\left(\frac{\pi+2k\pi}{3}\right)\right]$ for k = 0, 1, 2 thus the x are:

$x_1 = 2\left(cos\frac{\pi}{3} + isin\frac{\pi}{3}\right) = 2\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right) = 1 + i\sqrt{3}$

x_2 = 2(cosπ + isinπ) = 2(-1+0i) = -2

$x_3 = 2\left(cos\frac{5\pi}{3} + isin\frac{5\pi}{3}\right) = 2\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)$ = 1 - $i\sqrt{3}$

c) Let be u(x, y, z) any point of that plan then the vector $\vec{Pu}\begin{pmatrix}x-1\\y-0\\z+1\end{pmatrix}$ is orthogonal to $\begin{pmatrix}2\\2\\-1\end{pmatrix}$ and therefore 2(x-1) +2(y) – 1(z+1) = 0 is the equation of the plan finally we have 2x+2y-z-z = 0

d) f(x) = $\sqrt{x^2-4mx+5m-1}$

Dom f = $\{x \in \Re |\ x^2-4mx+5m-1 \geq 0\}$ by solving x^2-4mx+5m-1 we have x = 2m$\pm\sqrt{x^2-5m+1}$

x is real if $4m^2$-5m+1≥ i.e: m≥$\frac{1}{4}$ $or\ m \geq 1$

For m> $1\ or\ m > 1\ we\ have\ two\ distinct\ values\ of\ x\ and\ x^2-4mx+5m-1 \geq 0$

If x≤ $2-\sqrt{4m^2-5m+1}$ or If x≤ $2+\sqrt{4m^2-5m+1}$

Thus the domf = $\left\{\Re\ if\ \frac{1}{4} \leq m \leq 1\right]-\infty, 2m-\sqrt{4m^2-5m+1}\left[\cup\right]2m+\sqrt{4m^2-5m+1},\ \infty\left[\right\}$ for m<$\frac{1}{4}$ or ....

20. a) Calculate $\int \frac{sinx}{1+cosx}dx$ 2.5 marks

b) Let f(x) be a numerical function. Describe a removable discontinuity at a point x = a. Is there any point that illustrates this type of discontinuity given that $f(x) = \frac{x-2}{x^2-4}$? 5 marks

c) The table below summarizes the results of all the driving tests at a certain Test center during the first week of a certain month.

	Male	Female
Pass	32	43
Fail	8	15

i) A person is chosen at random from those who took a test that week

- Find the probability that the person passes the test
- Find the probability that it was a female who failed the test.

ii) A male is chosen. What is the probability that he did not pass the test? 7.5 marks

Answer:

a) $I = \int \frac{sinx}{1+cosx} dx = - \int \frac{d(1+cosx)}{1+cosx} = -ln(1+cosx) + C = \ln\frac{1}{1+cosx} + c \ for\ x \neq (2k+1)\pi$

Other method

$I = \int \frac{sinx}{1+cosx} dx \ let\ pose\ t = 1 + cosx \Leftrightarrow dt = d(1+cosx)$

dt = -sinx dx ⇒ sinx dx = -dt

$I = -\int \frac{dt}{t} = -ln|t| + C \ thus\ I = -ln|1 + cosx| + C$

b) **A function f(x) admits a discontinuity of the first type at point x = a if it is not continuous in x = a but admits the definite limit when x tends to a. If f is extendable for continuity in x =a.**

$f(x) = \frac{x-2}{x^2-4}$ $f\ is\ not\ continous\ at\ x = 2\ because\ 2 \notin Domf$

but $\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x-2}{x^2-4} = \frac{0}{0} IF$

$= \lim_{x\to 2} \frac{x-2}{(x-2)(x+2)} = \frac{1}{4}$

Thus f admits the discontinuity of the first type in x − 2

c) Let P be the person who passes successively the test, and F the person is female

	M	F	TOTAL
R	32	43	75
E	8	15	23
	40	58	98

i) $P(p) = \frac{75}{98}$

$P(\overline{P} \cap F) = P(\overline{P/F})\, P(F) = \frac{15}{58} . \frac{58}{98} = \frac{15}{98}$

ii) $P(\overline{P}/\overline{F}) = \frac{P(\overline{P}\cap\overline{F})}{P(\overline{F})} = \frac{8}{40} = \frac{1}{5}$

ADVENCED LEVEL MATHEMATICS NATIONAL EXAMINATION PAPER 2010 (MCB, MEG, MPC, MPG, PCM, PEM)

SECTION A: Attempt all questions. (55 marks)

01. In an arithmetical progression, the thirteenth term is 27 and the seventh is three times the second. Find the first term, the common difference and the sum of the first ten terms. 5 marks

Answer:

In arithmetical progression (AP)

Let a_1 be the first term and d be the common difference

$a_n = a + (n-1)d$

$a_{13} = a+(13-1)d \Leftrightarrow a_{13} = a+12d \Leftrightarrow a_{13}=27$

$a_7 = 3d_2$

$\Leftrightarrow a+(7-1)d = 3[a+(2-1)d]$

$\Leftrightarrow a+6d = 3(a+d)$

$6d-3d = 3a-a$

$3d = \frac{2a}{3}$

$a+12d = 27$

Again $a+12d = 27$

$a+12\left(\frac{2a}{3}\right) = 27$

$a+4(2a) = 27$

$a+8a = 27$

$9a = 27$

$a = 3$

$d = \frac{2a}{3} = \frac{2.3}{3} = 2$

$S_n = \frac{n}{2}(a+a_n)$

$S_n = \frac{n}{2}[a+a+(n-1)d] \Leftrightarrow S_n = \frac{n}{2}(2a+(n-1)d)$

$S_{10} = \frac{10}{2}(2.2 + (10-1)2) = 5(6+18) = 110$

02. Find the equation of the normal to the curve $2x^2-6xy+y^2=9$ in the point (4, 1). 5 marks

Answer:

The equation of the normal line to the curve is given

$N \equiv y - y_o = -\frac{1}{y'_o}(x-x_o)$ where P(4, 1) i.e $y_o=1$, $x_o = 4$

$y' = \frac{dy}{dx} \Leftrightarrow 2x^2-6xy+y^2 = 9$

$4x-6(y+x\frac{dy}{dx})+2y\frac{dy}{dx} = 0$

At the point (4, 1) we obtain :

$16-6(1+4\frac{dy}{dx})+2\frac{dy}{dx} = 0$

or $10-22\frac{dy}{dx}=0$ or $\frac{dy}{dx}=\frac{10}{22}=\frac{5}{11}=y'_0$

then the slope of the normal is $-\frac{1}{y'_0}=-\frac{11}{5}$ and the equation asked is $y-1=-\frac{11}{5}(x-4)$

or $y=1-\frac{11}{5}(x-4)$ or $y=\frac{49}{5}-\frac{11}{5}x$ or $11x-5y=49$

Other method

$2x^2-6xy+y^2=9$ (4, 1)

The Cartesian equation of the normal

$N\equiv y-y_0=\frac{f'_y(x_0-y_0)}{f'_x(x_0-y_0)}(x-x_0)$

$N\equiv y-y_0=\frac{6x_0-2y_0}{4x_0-6y_0}(x-x_0)$

$N\equiv y-1=\frac{-24+2}{16-6}(x-4)$

$N\equiv y-1=\frac{-22}{10}(x-4)$

$N\equiv y-1=\frac{-11}{5}(x-4)$

$N\equiv y=\frac{49}{5}-\frac{11}{5}x$

03. Two machines A and B produce 60% and 40% respectively of the total output of a factory. Of the parts produced by machine A, 3% are defective and of the parts produced by machine B, 5% are defective. A part is selected at random from a day's production and found to be defective. What is the probability that it came from machine A? 4 marks

Answer:

Let A be event that the part is produced by the machine P(A) = 60% = 0.6

Let B the event that the part is produced by machine P(B) = 40% = 0.4

Let D be the part which is defective $P(D)=P(A\cap D)+p$ we are required P(A\D)

$P(A\backslash D)=\frac{P(A\cap D)}{P(D)}=\frac{P(A)xP(D\backslash A)}{P(D)}=\frac{P(A)xP(D\backslash A)}{P(A\cap D)+P(B\cap D)}=\frac{P(A)xP(D\backslash A)}{P(A)xP(D\backslash A)+P(B)xP(D\backslash B)}=\frac{0.6x0.03}{0.6x0.03+0.4x0.05}=\frac{0.018}{0.018+0.020}=\frac{0.018}{0.038}=\frac{9}{19}$

Other way (Boyes's theorem)

$P(A\backslash D)=\frac{P(A)xP(D\backslash A)}{P(A).P(D\backslash A)+P(B).P(D\backslash B)}=\frac{0.6x0.03}{0.6x0.03+0.4x0.020}=\frac{18}{38}=\frac{9}{19}$

04. The amount A(t), in grams, of radioactive material in a sample after t years, is given by $A(t)=80(2^{-t/100})$.

a) Find the amount of material in the original sample. 1 mark

b) Calculate the half-life of the material. [The half-life is the time taken for half of the original material to decay]. 2 marks

c) Calculate the name taken for the material to decay to 1 gram. 2 marks

Answer:

a) The initial (original) sample $A(t)=80.(2^0)=80g$

b) For the half-life A(t) = 40

$\Rightarrow 40=80(2^{\frac{t}{100}})=\frac{1}{2}=2^{\frac{-t}{100}}\Leftrightarrow 2^{-1}=2^{\frac{-t}{100}}\Leftrightarrow\frac{-t}{100}=-1\Leftrightarrow t=100$

So the sample of the half life is 100

c) $A(t)=1\Leftrightarrow 80\left(2^{\frac{-t}{100}}\right)=1\Leftrightarrow 2^{\frac{-t}{100}}=\frac{1}{80}\Leftrightarrow\frac{t}{100}\log 2=\log 80$

$\Leftrightarrow t = \frac{100log80}{log2} = 632$

Or $\ln 2^{\frac{-t}{100}} = ln\frac{1}{80} \Leftrightarrow -\frac{t}{100}ln2 = -ln80 \Leftrightarrow t = \frac{100ln20}{ln2}$

then t = 632.19

05. Find the angle between the lines
$\frac{x+2}{2} = \frac{y+1}{2} = -2 \; and \; \frac{x+2}{3} = \frac{y}{6} = \frac{z-1}{2}$. 4 marks

Answer:

The vectors a = 2i+2j-k and b = 2i+6j-2k are parallel to two straight lines if
θ *is the cute angle between 2 lines then*

$\cos\theta = \frac{a.b}{......} = \frac{16}{3x7} = \frac{16}{21}$

thus $\theta = w^{-1}\left(\frac{16}{21}\right) = 0.704147414(rad) = 40.367$

06. Suppose that the profit P obtained in selling x units of a certain item each week is given by

$P = 50\sqrt{x} - 0.5x - 500$, $0 \le x \le 8000$.

Find the rate of change of P with respect to x when x = 1600. 2 marks

Answer:

The rate of change of p with respect to x is given by

$\frac{dp}{dx} = \frac{d}{dx}\left(50\sqrt{x} - 0.5x - 500\right) = \frac{25}{\sqrt{x}-0.5}$

At the point x = 1600, we have $\frac{dp}{dx} = \frac{25}{\sqrt{1600}} - 0.5 = \frac{5}{8} - \frac{1}{2} = \frac{1}{8}$

07. Find the intervals for which the following function is continuous:

$f(x) = \begin{cases} xsin\frac{1}{x}, x \neq 0 \\ 0, x = 0 \end{cases}$ 2 marks

Answer:

Since $y = \frac{1}{x}$ *is continuous except in the point* $x = 0$ *and that the function sin* is everywhere continuous. We need only to verify the continuity at the point x = 0

At that point $\lim_{x\to 0} xsin\frac{1}{x} = 0$

$$\lim_{x\to 0} xsin\frac{1}{x} = \lim_{x\to 0}\frac{sin\frac{1}{x}}{\frac{1}{x}} \quad let\ pose\ t = \frac{1}{x} \Leftrightarrow if\ x \to 0 \Leftrightarrow t = \infty$$

we have $\lim_{x\to 0}\frac{sin\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty}\frac{sint}{t} = 0$

then the function $f(x) = xsin\frac{1}{x}$ *is continuous on* $]-\infty, +\infty[$

Other method

$f(x) = \begin{cases} xsin\frac{1}{x}, x \neq 0 \\ 0, x = 0 \end{cases}$

Verify if f(x) is continuous at point $x_0 = 0$

1) $x_0 = 0 \in domf$

2) $f(x_0) = \lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^+} f(x)$

Let $t = \frac{1}{x}$ *if* $x \to 0 \Rightarrow t \to \infty$ *then* $\lim_{x \to \infty} \frac{\sin t}{t} = 0$ thus f(x) is continuous. i.e: $]-\infty, +\infty[$

08. Given the function $f(x) = 5 - \frac{4}{x}$, find all c in the interval [1, 4] such that $f'(c) = \frac{f(4)-f(1)}{4-1}$ 3 marks

Answer:

The angular coefficient of the secant which passes through (1, f(1)) and (4, f(4)) is $\frac{f(4)-f(1)}{4-1} = \frac{4-1}{4-1} = 1$

As f verifies the conditions of the theorem of LAGRANGE (AVERAGE VALUE THEOREM)

There is at least one value on the interval $]1, 4[$ *such that* $f'(c) = 1$

Solving the equation: f'(c)= 1 we have $f'(c) = \frac{4}{c^2} = 1 \Rightarrow 4 = c^2 \Rightarrow c = \pm 2$

Finally on the interval (1, 4) we take c = 2

09. Find the volume of the solid formed by revolving the region bounded by the graphs of $y=x^2+1$, y=0, x=0 and x=1 about the y-axis. 4 marks

Answer:

By the disc method, two integrals are necessary for finding the volume.

$$V = \pi\int_0^1 (1^2 - 0^2)dy + \pi\int_1^2 (1^2 - \sqrt{4-1})dy = \pi\int_0^1 1dy + \pi\int_1^2 (2-y)dy$$

$$= [\pi y]_0^1 + \pi\left[2y - \frac{y^2}{2}\right]_1^2 = \pi\left[1 + y - 2 - 2 + \frac{1}{4}\right] = \frac{3}{2}\pi\ UV$$

Or again

$$V = \pi\int_0^1 1^2 dy - \pi\int_1^2 \left(\sqrt{y^2-1}\right)^2 dy = \pi\int_0^1 dy - \pi\int_1^2 (y^2-1)dy$$

$$= [\pi y]_0^2 - \pi\left[\frac{y^2}{2} - y\right]_1^2 = \pi(2-0) - \pi[(2-2)-(\frac{1}{2}-1)) = 2\pi - \frac{\pi}{2} = \frac{3\pi}{2}\ UV$$

Other method

From the formula

$$V = 2\pi\int_a^b xydx \text{ we have } \Rightarrow V = 2\pi\int_0^1 x(1+x^2)dx \quad \Rightarrow V = 2\pi\int_0^1 (x^3+x)dx$$

$$V = 2\pi\left[\frac{x^4}{4} + \frac{x^2}{2}\right]_0^1 = 2\pi\left(\frac{1}{4} + \frac{1}{2}\right) = 2\pi\left(\frac{3}{4}\right) = \frac{3\pi}{2}\ UV$$

10. Prove that the function f: IR → IR: x → $f(x) = x^2$ is neither an injection nor a surjection. 2 marks

Answer:

i) Consider the element 4 ∈ ℜ *the image set.*

Then the equation

f(x) = 4, has two solutions x = -2, 2∈ℜ the domain of definition.

Therefore f is not injective

ii) Consider the element -4 ∈ℜ the image set, therefore the equation f(x) = -4 is not the solution x ∈ℜ, the domain of definition, f is not surjective.

Other method

i) $f(x) = x^2$, f(x) is surjective if and only if $\forall x, y \in \Re, f(x) = f(y) \Rightarrow x = y$

Therefore $f(x) = f(y) \Leftrightarrow x^2 = y^2 \Rightarrow x = \pm y$

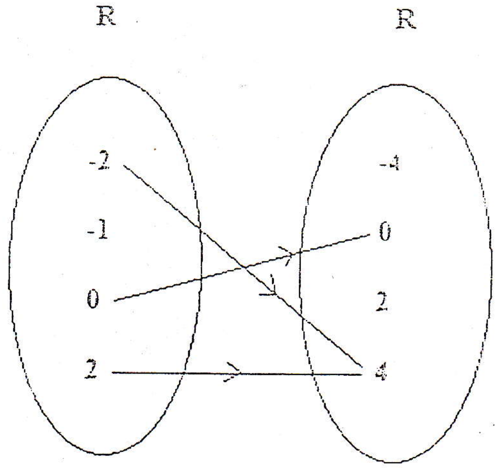
Domain is not injective

ii) f(x) is surjective if and only if $\forall x, y \in \Re, such\ that\ f(x) = y$

Therefore $x^2 = y \Rightarrow x = \pm\sqrt{y}$ i.e $y \geq 0$

Thus Im f = $\Re^f \neq \Re$

Or again



RMK $f(x) = x^2$ is not injective because it is even.

11. Evaluate the following limit: $\lim_{x\to\infty}\prod_{k-1}^{\infty}(1+\frac{k}{n^2})$. 4 marks

Answer:

Let pose

$y = \lim_{x\to\infty}\prod_{k=1}^{\infty}\left(1+\frac{k}{n^2}\right)$

then $y = \lim_{x\to\infty}\prod_{k=1}^{\infty}\left(1+\frac{k}{n^2}\right) \Rightarrow lny = ln\left[\prod_{k=1}^{\infty}\left(1+\frac{k}{n^2}\right)\right]$

$= \lim_{x\to\infty}\ln\left(\prod_{k=1}^{\infty}\left(1+\frac{k}{n^2}\right)\right) = \lim_{x\to\infty}\sum_{k=1}^{\infty}\ln\left(1+\frac{k}{n^2}\right) = \sum_{k=1}^{\infty}\lim_{x\to\infty}\ln\left(1+\frac{k}{n^2}\right)$

As $\ln\left(\frac{1+x}{x}\right) = 1 \Rightarrow \lim_{x\to\infty}\ln\left(\frac{1+\frac{k}{n^2}}{\frac{k}{n^2}}\right) = 1$ and $\frac{k}{n^2} = 0$ $when\ n \to \infty$

Then $\ln y = \sum_{k=1}^{\infty}\lim_{x\to\infty}\ln\left(1+\frac{k}{n^2}\right) = \sum_{k=1}^{\infty}\lim_{x\to\infty}\left[\frac{k}{n^2}\ln\frac{\left(1+\frac{k}{n^2}\right)}{\frac{k}{n^2}}\right] =$

$\sum_{k=1}^{\infty}\lim_{x\to\infty}\frac{k}{n^2}\lim_{x\to\infty}\ln\frac{\left(1+\frac{k}{n^2}\right)}{\frac{k}{n^2}}$

$= \sum_{k=1}^{\infty}\lim_{x\to\infty}\frac{k}{n^2} = 0$

$= \lim_{x\to\infty}\frac{1}{n^2}\sum_{k=1}^{\infty}k = \frac{n(n+1)}{2n^2} = \frac{1}{2}$

$lny = \frac{1}{2} \Leftrightarrow y = e^{\frac{1}{2}}$

12. Evaluate the integral $\int_{-1}^{1}\frac{dx}{x^2-2xcos\alpha+1}$ $(0<\alpha<\pi)$ 4 marks

Answer:

$\int_{-1}^{1}\frac{dx}{x^2-2xcos\alpha+1} = \int_{-1}^{1}\frac{dx}{(x-cos\alpha)^2-cos^2\alpha+1} = \int_{-1}^{1}\frac{dx}{(x-cos\alpha)^2+sin^2\alpha} = \int_{-1-cos\alpha}^{1-cos\alpha}\frac{dx}{(t)^2+sin^2\alpha}$

$= \frac{1}{sin\alpha}\left[arctan\frac{t}{sin\alpha}\right]_{-1-cos\alpha}^{1-cos\alpha} = \frac{1}{sin\alpha}\left[arctan\frac{1-cos\alpha}{sin\alpha} + arctan\frac{1+cos\alpha}{sin\alpha}\right]$

$= \frac{1}{sin\alpha}\left[arctan(tan\frac{\alpha}{2}) + arctan(cos\frac{\alpha}{2})\right] = \frac{1}{sin\alpha}\left[arctan(tan\frac{\alpha}{2}) + arctan(\frac{\pi}{2}-\frac{\alpha}{2})\right]$

$= \frac{1}{sin\alpha}\left[\frac{\alpha}{2}+\frac{\pi}{2}-\frac{\alpha}{2}\right] = \frac{\pi}{2sin\alpha}$

Other method

$\int_{-1}^{1}\frac{dx}{x^2-2xcos\alpha+1} =$

$\Rightarrow x^2 - 2xcos\alpha + 1 = 0$

$\Delta = 4\ cos^2\alpha - 4 = -4(1 - cos^2\alpha) = -4sin^2\alpha$

$\sqrt{-\Delta} = 2sin\alpha\ with\ \Delta' < 0$

$we\ have\ \int\frac{dx}{ax^2+bx+c} = \frac{2}{\sqrt{-\Delta}}arctan\frac{(ax^2+bx+c)'}{\sqrt{-\Delta}}$

$\Rightarrow \int_{-1}^{1}\frac{dx}{x^2-2xcos\alpha+1} = \left[\frac{2}{2sin\alpha}arctan\frac{2x-2cos\alpha}{2sin\alpha}\right]_{-1}^{1} = \left[\frac{1}{sin\alpha}arctan\frac{x-cos\alpha}{sin\alpha}\right]_{-1}^{1}$

$= \frac{1}{sin\alpha}\left[arctan\frac{1-cos\alpha}{sin\alpha} + arctan\frac{1+cos\alpha}{sin\alpha}\right]$

13. Show the polynomial $T_m(x) = \frac{1}{2^{m-1}}(m\ arccosx)$ (m=1, 2, 3,…..), satisfies the following differential equation:

$(1-x^2)T''_m(x) - xT'_m(x) + m^2T_m(x) = 0$. 3 marks

Answer:

$T'_m(x) = \frac{m}{2^{m-1}}\frac{sin\ m\ arc(cosx)}{\sqrt{1-x^2}}$

$T'_m(x) = -\frac{m\ cos(m\ arccosx)}{2^{m-1}1-x^2} + \frac{xm\ sin(m\ arccosx)}{2^{m-1}(1-x^2)^{\frac{3}{2}}} = -\frac{m}{1-x^2}T_m(x) + \frac{x}{1-x^2}T'_m(x)$

Putting the values of $T_m(x), T'_m(x)$, *and* T''_m *in the given equation*

We obtain: $(1-x^2)T''(x) - xT'_m(x) + m^2T_m(x) = -m^2T_m(x) + xT_m(x) - xT'_m(x) + m^2T_m(x) = 0$

14. a) Express $\delta = sin^2t - 2(1 - cost)$ in terms of $sin\frac{t}{2}$ $(t\epsilon IR)$. 2.5 marks

b) Solve the equation 2u(1-cost)+2usint+1 = 0, uϵZ. 3 marks

Answer:

a) $sin^2t = (sint)^2 = \left(2sin\frac{t}{2}\ cos\frac{t}{2}\right)^2 = 4sin^2\frac{t}{2}\ cos^2\frac{t}{2}$

$1-cost = cos^2\frac{t}{2} + sin^2\frac{t}{2} - \left(cos^2\frac{t}{2} - sin^2\frac{t}{2}\right) = 2\ sin^2\frac{t}{2}$

$\delta = sin^2t - R(1 - cost) = 4sin^2\frac{t}{2}\ cos^2\frac{t}{2} - 4sin^2\frac{t}{2} = 4sin^2\frac{t}{2}\left(cos^2\frac{t}{2} - 1\right) =$

$4sin^2\frac{t}{2}\left(-sin^2\frac{t}{2}\right) = -4sin^4\frac{t}{2}$

b) 2u(1-cost) – 2u sin t +1 =0

2u[(1-cost)-sint] = -1

$2u = \frac{1}{1-cost-sint} \Rightarrow u = \frac{1}{2(cost+sin\,t-1)}$

Solving with respect to t

2u(1-cost)-2u sin t + 1 =0

2u(1-cost-sin t) = -1

$1-cost-sint = -\frac{1}{2u}$

$\cos t + \sin t = \frac{1+2u}{2u}$

15. Suppose that a≡b (mod m) and c≡d (mod m).
Prove that:
i) a+c ≡b+d (mod m) 1.5 marks
ii) ac ≡ bd (mod m) 1.5 marks
Answer:
i) For an integer q_1 and q_2 we have
$a = b+mq_1$ and $c = d+mq_2$
then $a+c = b+mq_1 +d+mq_2$
$= b+d+m(q_1+q_2)$
$= b+d+mq_3$
$= b+d(\text{mod } m)$
ii) $a.c = (b+mq_1)(d+mq_2)$
$= bd + m(q_1d+q_2b+mq_1q_2)$
$= bd+mq_4$, q_4 is an integer
$= bd(\text{mod } m)$

SECTION B: Attempt any three questions. (45 marks)

16. a) At noon, two boats P and Q are at points whose position vectors are 4i+8j and 4i+3j respectively. Both boats are moving with constant velocity; the velocity of P is 4i+j and the velocity of Q is 2i+5j, (all distances are in kilometers and the time is measured in hours). Find the position vectors of P, Q and $\overline{PQ}$ after t hours and hence express the distance PQ between the boats in terms of t. Find the least distance between the boats. 7.5 marks

b) The random variable X has the following probability distribution:

X	P(X=x)
2	A
4	$2a^2$-a
6	a^2+a-1

Find:
a) the value of a. 3 marks
b) E(X) 1 mark
c) V(X) 2 marks
d) SD(X) 1.5 marks

Answer:
a) After t hours, the displacement of P from its starting point is
$t(4\vec{i}+\vec{j})$ thus $P = (4\vec{i}+3\vec{j}) + t(4\vec{i}+\vec{j})$
$= (4+4t)\vec{i} + (8+t)\vec{j}$.
Similarly $Q = (4+2t)\vec{i} + (2\vec{i}+\vec{j})t$
$= (4+2t)\vec{i} + (3+5t)\vec{j}$
Then $\overline{PQ} = Q - P = [(4+2t)\vec{i} + (3+5t)\vec{j}] - [(4+4t)\vec{i} + (8+t)\vec{j}]$
$\overline{PQ} = -2t\vec{i} + (-5+4t)\vec{j}$
$\overline{PQ}^2 = (-2t)^2 + (-5+4t)^2 = 20t^2 - 40t + 25$
The distance between two boat sis given by $\sqrt{20t2 - 40t + 25}km$
For the smallest distance, let consider:

$\overline{PQ}^2 = 20t^2 - 40t + 25 = 20(t^2 - 25t + 1) + 5 = 20(t-1)^2 + 5$

As $(t-1)^2$ cannot be negative, its smallest value is zero and it is obtained for t = 1

then the smallest value of $\overline{PQ}^2$ is 5.

The possible shortest distance between two boats is then $\sqrt{5}$km.

b) i) the given distribution is the probability distribution S if

$0 \le P(x=x) \le 1$ and $\sum P(x=x) = 1$ *then* $\sum P(x=x) = 1$

$\Rightarrow a^2 + (a^2 - a) + (a^2 + a - 1) = 1$

$3a^2 + a - 2 = 0$

$(3a-2)(a+1) = 0 \Leftrightarrow a = -1, \frac{2}{3}$

$\frac{2}{3}$ *is the one possible value because a cannot be negative*

X	P(x=x)	Xp(x=x)	$x^2P(x=x)$
2	2/3	4/3	8/3
4	2/9	8/3	32/9
6	1/9	6/9	36/9
	9/9	26/9	92/9

ii) $E(x) = \sum xP(x=x) = \frac{26}{9}$

iii) $V(x) = E(x^2) - E^2(x) = \sum x^2P(x) = 1 - E^2(x) = \frac{92}{9} - \left(\frac{26}{9}\right)^2 = \frac{152}{81}$

iv) $SD(x) = \sqrt{V(x)} = \sqrt{\frac{152}{81}} = 1.3698$

17. a) Find the shortest distance between the skew lines

$L_1 = \begin{cases} x = 5 + 2\lambda \\ y = 3 - \lambda \\ z = 0 \end{cases}$ and $\begin{cases} x = 2 \\ y = \mu \\ z = 9 - \mu \end{cases}$ 9 marks

b) If $A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 2 & -1 & -1 \end{bmatrix}$, evaluate A^3 and hence find A^{-1}. 7 marks

Answer:

a) The shortest distance between the two points one on each of two given straight line is given by

$|\overline{PQ}| = \frac{|AB.W|}{\|W\|} = \frac{|(b-a).(uxw)|}{\|uxv\|}$

As $u = 2\vec{i} + \vec{j}$ is parallel to the first line and $\vec{v} = \vec{j} - \vec{k}$ to the second

$w = uxv = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 0 \\ 0 & 1 & -1 \end{vmatrix} = \vec{i} + 2\vec{j} + 2\vec{k}$

$\Leftrightarrow |w| = \sqrt{1^2 + 4^2 + 4^2} = 3$

A = (5, 3, 0) lies on the first line and B(2, 0, 9) on the second.

Now $|\overline{AB}| = B - A = -3\vec{i} - 3\vec{j} + 9\vec{k}$ then

$|\overline{AB}.\overline{W}| = -3 - 6 + 18 = 9$

The asked distance is $\frac{|AB.W|}{|W|} = \frac{9}{3} = 3$

b) $A^2 = \begin{pmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 2 & -1 & -2 \end{pmatrix}\begin{pmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 2 & -1 & -2 \end{pmatrix} = \begin{pmatrix} 1+2-4 & 2+2+2 & -2+2+4 \\ 1+1+2 & 2+1-1 & -2+1-2 \\ 2-1-4 & 4-1+2 & -4-1+4 \end{pmatrix}$

$= \begin{pmatrix} -1 & 6 & 4 \\ 4 & 2 & -3 \\ -3 & 5 & -1 \end{pmatrix}$

$A^3 = \begin{pmatrix} -1 & 6 & 4 \\ 4 & 2 & -3 \\ -3 & 5 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 2 & -1 & -2 \end{pmatrix} = \begin{pmatrix} 13 & 0 & 0 \\ 0 & 13 & 0 \\ 0 & 0 & 13 \end{pmatrix}$

$A^3 = 13I$, it gives $A\frac{1}{13}A^2 = I \Leftrightarrow A^{-1} = \frac{I}{13}A^2 = \frac{I}{13}\begin{pmatrix} -1 & 6 & 4 \\ 4 & 2 & -3 \\ -3 & 5 & -1 \end{pmatrix}$?

18. Find the particular solution to the following differential equation, which satisfies the given initial values; $y''-2y'=x+2e^x$; $y(0) = 0$; $y'(0) = 1$ 15 marks

Answer:

The characteristic equation is $k^2-2k=0$ has two solutions $k = 0$ or $k = 2$

Thus the general solution of the homogeneous equation is

$y_h = c_1+c_2e^{2x}$

As $f(x) = x+2e^x$ our first choice of y_{p1} *could be:*

$(A+Bx) +ce^x$

But, as zero is the root of the characteristic equation, we are going to multiply the polynomial part by x, we have then

$y_{p1} = Ax+Bx^2+c\,e^x$

$y'_{p1} = A+2Bx+c\,e^x$

$y''_{p1} = 2B+c\,e^x$

Replace in the given equation, we obtain

$(2B+c\,e^x) -2(A+2Bx+c\,e^x) = x+2e^x$

$(2B-2A) -4Bx-c\,e^x = x+2e^x$

Equalizing the coefficients of the same term, we have :

$\begin{cases} 2B-2A=0 \\ -4B=1 \\ -c=2 \end{cases} \Leftrightarrow \begin{cases} A=B=\frac{1}{4} \\ c=-2 \end{cases}$

Then $y_{p1} = -\frac{1}{4}x - \frac{1}{4}y^2 - 2e^x$

And the general solution is

$y = c_1+c_2e^{2x} - \frac{1}{4}x - \frac{1}{4}y^2 - 2e^x$

$y(0) = 0 \Leftrightarrow c_1+c_2-2=0$

$y'(x) = 2e^{2x}\frac{1}{4} - \frac{1}{2}x - 2e^2$

$\Leftrightarrow y'(0) = 1 \Leftrightarrow 2c_2-\frac{1}{4} - 2 = 1$

$c_2 = \frac{13}{8}$

$c_1+c_2-2 = 0 \Rightarrow c_1 = \frac{3}{8}$

Then the asked particular solution is

$y_{p1} = \frac{3}{8} + \frac{13}{8}e^{2x} - \frac{1}{4}x - \frac{1}{4}x^2 - 2e^x$

19. a) Sketch the graph of the polar equation

r$=\frac{-32}{3-5sin\theta}$ 7.5 marks

b) Determine the convergence or divergence of

$\sum_{n=1}^{\infty}\frac{n^3}{3^n}$. 3 marks

Answer:

a) $r=\frac{-32}{3-5sin\theta}$

In cartesian coordinates we know $r=\sqrt{x^2+y^2} \Leftrightarrow sin\theta=\frac{y}{\sqrt{x^2+y^2}}$

Then we obtain $3\sqrt{x^2+y^2}=5y-32$

Squaring two sides, we obtain

$9(x^2+y^2)= 25y^2-2-320y+1024$

Or $9x^2-16y^2+320y-1024=0$

$9x^2-16(y^2-20y)-1024=0$

Or $9x^2-16(y-10)^2=-576$

$16(y-10)^2-9x^2=576$

$16(y-10)^2-9x^2=576$

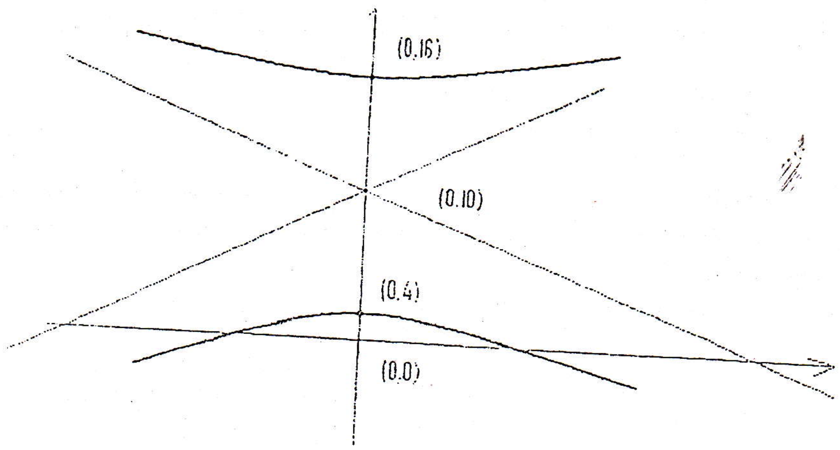
Or $\frac{(y-10)^2}{16}=\frac{x^2}{64}=1$ which is an hyperbola whose conjugate axis is OY

$a^2= 36 \Leftrightarrow a=6$

$b^2= 64 \Leftrightarrow b=8$

Centre C(0, 10)

Vertex $S_1(0, 16)$ and $S_2(0, 4)$



b) D'alembert theorem

$\lim_{n\to\infty}\frac{U_n+1}{U_n}<1 \rightarrow converges$

$\lim_{n\to\infty}\frac{U_n+1}{U_n}>1 \rightarrow Diverges$

$\lim_{n\to\infty}\frac{U_n+1}{U_n}=1 \rightarrow No\ conclusion$

$U_n=\frac{(n)^3}{3^n}$

$U_{n+1}=\frac{(n+1)^3}{3^{n+1}}$

$$\frac{U_{n+1}}{U_n} = \frac{(n+1)^3}{3^{n+1}} : \frac{(n)^3}{3^n} = \frac{(n+1)^3}{3^{n+1}} x \frac{3^n}{(n)^3} = \frac{(n+1)^3}{3n^3}$$

$$\lim_{n\to\infty} \frac{U_n+1}{U_n} = \lim_{n\to\infty} \frac{(n+1)^3}{3n^3} = \frac{1}{3}\lim_{n\to\infty} \frac{n^3+3n^2+3n+1}{n^3} = \frac{1}{3}\lim_{n\to\infty} \frac{n^3(1+\frac{3}{n}+\frac{3}{n^2}+\frac{1}{n^3})}{n^3} = \frac{1}{3}$$

$\lim_{n\to\infty} \frac{U_n+1}{U_n} = \frac{1}{3} < 1$ converges

20. Let $A = \begin{bmatrix} 5 & 2 & -2 \\ 2 & 5 & -1 \\ -2 & -2 & 5 \end{bmatrix}$

a) Verify that $\det(\lambda I_3 - A)$, the characteristic polynomial of A, is given by $(\lambda\text{-}3)^3(\lambda\text{-}9)$. 4 marks

b) Find a non-singular matrix P such that $P^{-1}AP$= diag(3, 3, 9). 11 marks

Answer:

a) $\text{DET}(\lambda I - A)$

$$p(\lambda) = \begin{vmatrix} \lambda-5 & -2 & 2 \\ -2 & \lambda-5 & 2 \\ 2 & 2 & \lambda-5 \end{vmatrix}$$

$\Rightarrow (\lambda-5)\begin{vmatrix} \lambda-5 & 2 \\ 2 & \lambda-5 \end{vmatrix} + 2\begin{vmatrix} -2 & 2 \\ 2 & \lambda-5 \end{vmatrix} + 2\begin{vmatrix} -2 & \lambda-5 \\ 2 & 2 \end{vmatrix}$

$\Leftrightarrow (\lambda-5)((\lambda-5)^2-4) + 2(2+2\lambda+5) + 2(-2\lambda+6)$

$\Leftrightarrow (\lambda-5)(\lambda-5-2)(\lambda-5+2) + 4(-2\lambda+6)$

$\Leftrightarrow (\lambda-5)(\lambda-7)(\lambda-3) - 8(\lambda-3)$

$\Leftrightarrow (\lambda-3)[(\lambda-5)(\lambda-7)-8] \Leftrightarrow (\lambda-3)(\lambda^2-12\lambda+35) \Leftrightarrow (\lambda-3)(\lambda-3)(\lambda-9)$

So $\det(\lambda I - A) \neq (\lambda-3)^2(\lambda-9)$

The eigen values of the matrix A are:

$\lambda_1 = \lambda_2 = 3, \lambda_3 = 9$

b) The matrix P is given by the juxtaposition of eigenvectors associated to the different eigenvalues. To find eigenvectors associated to eigenvalues, we solve now the equation:

$(\lambda I - A)X = 0$ *where X is the column vector*

$\lambda = 3$

$$\begin{pmatrix} 3-5 & -2 & 2 \\ -2 & 3-5 & 2 \\ 2 & 2 & 3-5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -2 & -2 & 2 \\ -2 & -2 & 2 \\ 2 & 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

2x-2y+2z = 0 ⇔ -x-y+z = 0 ⇔ z = x+y

Let pose x = r, y = s r and s ∈ℜ

We have $V = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ x+y \end{pmatrix} = \begin{pmatrix} r \\ s \\ r+s \end{pmatrix} = \begin{pmatrix} r \\ 0 \\ r \end{pmatrix} + \begin{pmatrix} 0 \\ s \\ s \end{pmatrix} = r\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + s\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Then the two eigenvectors associated to the eigenvalue $\lambda = 3$ *are*:

$V_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ $V_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$\lambda = 9$

$$\begin{pmatrix} 9-5 & -2 & 2 \\ -2 & 9-5 & 2 \\ 2 & 2 & 9-5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 4 & -2 & 2 \\ -2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 4x - 2y - 2z = 0 \\ -2x + 4y + 2z = 0 \\ 2x + 2y + 4z = 0 \end{cases} \Leftrightarrow \begin{cases} 6y + 6z = 0 \\ 6x + 6z = 0 \end{cases} \Leftrightarrow \begin{cases} y = -z \\ x = -z \end{cases}$$

Let pose t = z $t \in \Re$

$$V = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -t \\ -t \\ t \end{pmatrix} = t\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

$$V_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

Thus the asked matrix P is :

$$P = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

P is not singular because det P = 3

ADVENCED LEVEL MATHEMATICS NATIONAL EXAMINATION PAPER 2011 (MCB, MCE, MEG, MPC, MPG, PCM, PEM)

SECTION A: Attempt all questions. (55 marks)

01. What values (real numbers) of x satisfying the following condition:
   a) 4(x+5) – 6(2x+3) = 3(x+14) – 2(5-x) + 9 **2.5 marks**
   b) $|6 - 3x| > 14$ where $|a|$ stands for absolute value of a defined as $|a| = \begin{cases} a \text{ if } a \geq 0 \\ -a \text{ if } a \leq 0 \end{cases}$ 2 marks

   Answer:

   a) 4(x+5)-6(2x+3) = 3(x+14)-2(5-x)+9
   ⇔ 4x+20-12x-18 = 3x+42-10+2x+9
   ⇔ -8x+2 = 5x+41
   ⇔ -8x-5x = -2+41
   ⇔ - 13x = 39
   ⇔ x = -3
   S = {−3}

   b) $|6 - 3x| > 14 \Leftrightarrow 6 - 3x > \text{ or } 6 - 3x < -14$
   $\Leftrightarrow -3x > 8 \text{ or } -3x < -20 \Leftrightarrow x < -\frac{8}{3} \text{ or } x > \frac{20}{3}$
   $S = \left]-\infty, -\frac{8}{3}\right[\cup \left]\frac{20}{3}, +\infty\right[$

02. If (u, v, w) is a basis of the real vector space $\Re^3$ determine whether or not (u+v, u+2w, u-w) is also a basis of $\Re^3$. 3 marks.

   Answer:

   Let a, b and c be any three real number such that a(u+v) + b(u+2w) +c(u-w) = 0
   We have:
   (a+b+c) u +av + (2b-c) w = 0

Since (u, v, w) is a basis of $\Re^3$, $\begin{cases} a+b+c=0 \\ a=0 \\ 2b-c=0 \end{cases}$

The system has a = 0, b = 0, c = 0 as solution.

Therefore (u+v, u+2w, u-w) is also a basis of $\Re^3$.

03. Let $f: \Re \rightarrow \Re: f(x) = \frac{x^2-1}{|x-1|}$

a) Find $\lim_{x\to1>} f(x)$ and $\lim_{x\to1<} f(x)$ 3.5 marks

b) Discuss the limit of $f(x)$ as x approaches 1. 0.5 mark

c) Sketch the graph of $f(x)$. 1 mark

Answer:

a) $f(x) = \frac{x^2-1}{|x-1|} = \begin{cases} \frac{x^2-1}{x-1} \; if \; x > 1 \\ \frac{x^2-1}{-x+1} \; if \; x < 1 \\ undefined \; if \; x = 1 \end{cases}$

Let $x \neq 1$

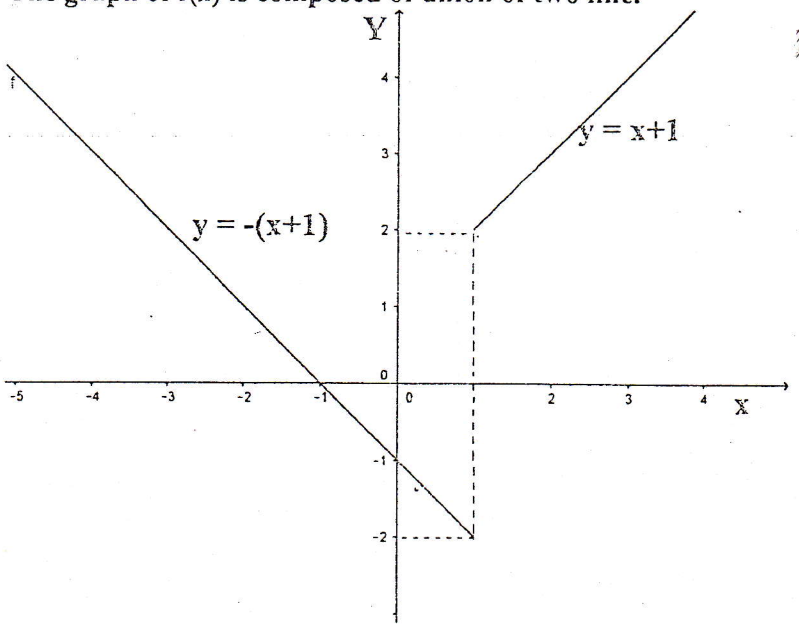
$f(x) = \frac{x^2-1}{|x-1|} = \begin{cases} \frac{x^2-1}{x-1} \; if \; x > 1 \\ \frac{x^2-1}{-x+1} \; if \; x < 1 \end{cases}$

Therefore: i) $\lim_{x\to1_>} f(x) = \lim_{x\to1_>} (x+1) = 2$

ii) $\lim_{x\to1_<} f(x) = \lim_{x\to1_<} -(x+1) = -2$

b) Since $\lim_{x\to1_>} f(x) \neq \lim_{x\to1_<} f(x), \lim_{x\to1} f(x)$ *does not exist.*

c) The graph of f(x) is composed of union of two line.



04. Express $f(x)=\frac{3x^3-x^2-13x-13}{x^2-x-6}$ in partial fractions. Then find antiderivative of $f(x)$. 4.5 marks

Answer:

By long division $f(x)=\frac{3x^3-x^2-13x-13}{x^2-x-6}=3x+2+\frac{7x-1}{x^2-x-6}$

$$\frac{7x-1}{x^2-x-6}=\frac{7x-1}{(x-3)(x+2)}=\frac{A}{x-3}+\frac{B}{x+2}=\frac{A(x+2)+B(x-3)}{(x-3)(x+2)}$$
$$=\frac{Ax+2A+Bx-3B}{(x-3)(x+2)}=\frac{(A+B)x+2A-3B}{(x-3)(x+2}$$

$\frac{7x-1}{x^2-x-6}=\frac{(A+B)x+2A-3B}{(x-3)(x+2)}\Leftrightarrow 7x-1=(A+B)x+2A-3B$

$\Leftrightarrow A+B=7$

$2A-3B=-1$

$\Leftrightarrow A=4$ and $B=3$

$\frac{7x-1}{(x-3)(x+2)}=\frac{4}{x-3}+\frac{3}{x+2}$

In partial fraction $(x)=\frac{3x^3-x^2-13x-13}{x^2-x-6}=3x+2+\frac{4}{x-3}+\frac{3}{x+2}$

Antiderivative of f(x)

$$\int f(x)dx=\int\left(3x+2+\frac{4}{x-3}+\frac{3}{x+2}\right)dx=$$
$$\int(3x+2)dx+\int\left(\frac{4}{x-3}\right)dx+\int\left(\frac{3}{x+2}\right)dx=3\int xdx+2\int dx+4\int\frac{dx}{x-3}+3\int\frac{dx}{x+2}$$
$$\int f(x)dx==3\frac{x^2}{2}+2x+4ln|x-3|+3\ln|x+2|+C$$

05. Find the number of ways that 6 teachers can be assigned to 4 sections of mathematics course if no teacher is assigned to more than one section. 2.5 marks

Answer:

The total number in simple way is = 6x5x4x3 = 360

Or

$P=\frac{6!}{(6-4)!}=\frac{6x5x4x3x2!}{2!}=6x5x4x3=360$

06. In Euclidian space, find an equation for the plane consisting of all points that are equidistant from the points (-4, 2, 1) and (2, -4, 3). 3 marks

Answer:

$$\alpha_o=\begin{vmatrix}x & -4 & 2\\ y & 2 & -4\\ z & 1 & 3\end{vmatrix}=0$$

The equation is 9x+15y-12z =0 ⇔ 3x+5y-4z =0

07. Find any asymptotes of the function f if $f(x)=\frac{x-1}{x^2-1}$ 3 marks

Answer:

Asymptotes:

HA:

$\lim_{x\to\pm\infty}\frac{x-1}{x^2-1}=\frac{\infty}{\infty}I.F$

$$\lim_{x\to\pm\infty}\frac{x-1}{x^2-1}=\lim_{x\to\pm\infty}\frac{x(1-\frac{1}{x})}{x^2(1-\frac{1}{x^2})}=\lim_{x\to\pm\infty}\frac{1}{x}=0$$

$H.A\equiv y=0$

V.A:

$f(x)=\frac{x-1}{x^2-1}=\frac{x-1}{(x-1)(x+1)}=\frac{1}{x+1}$

$\lim_{x\to -1}\frac{1}{x+1}=\infty$

V.A $\equiv$ x = -1

No O.A

08. Find a second degree polynomial P(x) such that P(2) = 5, P'(2)= 3 and P''(2) =2 where P' and P''(2) =2 where P' and P'' are first and second derivatives of P respectively. 4 marks

Answer:

Second degree polynomial P(x) = ax^2+bx+c

P(2) = $a(2)^2+2b+c = 2 \Leftrightarrow 4a+2b+c = 2$

P(x)'= 2ax+ b $\Leftrightarrow$ P(2)' = 2.a.2 + b = 4a+b = 3

P(x)'' = 2a = 2

a = 1

4a+b = 3$\Leftrightarrow$ 4x1+b = 3 $\Leftrightarrow$ b = -1

4a+2b+c = 2$\Leftrightarrow$ 2x1+2(-1) + c = 2$\Leftrightarrow$ 2-2+c = 2 $\Leftrightarrow$ c = 2

So a = 1, b = -1 and c = 2

So the polynomial is x^2-x+2

09. a) Evaluate the derivative of $f(x)=\ln(x^3+7x^2)$ where ln stands for natural logarithm function; 1.5 marks

b) and evaluate the integral $\int\frac{x^3}{x^4+7}dx$ 1.5 marks

Answer:

a) $f(x)'=\ln(x^3+7x^2)'=\frac{x^3+7x^2}{(x^3+7x^2)}=\frac{3x^2+14x}{(x^3+7x^2)}=\frac{(3x+14)}{x(x+7)}$

b) $\int\frac{x^3}{x^4+7}dx$

Let $t=x^4+7 \Leftrightarrow dt=4x^3dx \Leftrightarrow x^3dx=\frac{dt}{4}$

$\frac{1}{4}\int\frac{dt}{t}=\frac{1}{4}ln|t|+C=\frac{1}{4}ln|x^4+7|+C$

10. How many distinct permutations can be made from the letters of the word ''infinity''? 2 marks

Answer:

$P=\frac{8!}{3!2!}=\frac{8x7x6x5x4x3x2x1}{3x2x1x2x1}=\frac{40320}{12}=3360$

11. Find the domain and the derivative of the numerical function f if $f(x)=\frac{x}{1-\ln(x-1)}$. 4 marks

Answer:

Domain $1-\ln(x-1)\neq 0$ ln(x-1) $\neq 1 \Leftrightarrow$ x-1 $\neq e \Leftrightarrow x\neq e+1$

and x-1> 0$\Leftrightarrow$ x > 1

Domain : $]1,e+1[\cup]e+1,+\infty[$

$f(x)'=\frac{1-ln(x-1)-x\left(-\frac{1}{x-1}\right)}{[1-ln(x-1)]^2}=\frac{1-ln(x-1)+\frac{x}{x-1}}{[1-ln(x-1)]^2}=\frac{[1-ln(x-1)](x-1)+x}{[1-ln(x-1)]^2}=\frac{x-1-(x-1)ln(x-1)+x}{[1-ln(x-1)]^2}$

$=\frac{(x-1)ln(x-1)+2x-1}{[1-ln(x-1)]^2}$

12. The probability that a patient recovers from a delicate heart operation is 0.8. What is the

probability that

a) Exactly 2 of the next 3 patients who have this operation survive? **1.5 marks**

b) All of the next 3 patients who have this operation survive? **1.5 marks**

Answer:

Let S "the patient recovers"; $p(S) = 0.8$

a) $p(2 \text{ of the next 3 patients recover}) = \binom{3}{2}(0.8)^2(0.2) = \frac{3!}{2!1!}(0.8)^2(0.2) =$
$3(0.8)^2(0.2) = 0.384$

b) $p(3 \text{ patients recover}) = \binom{3}{3}(0.8)^3 = (0.8)^3 = 0.512$

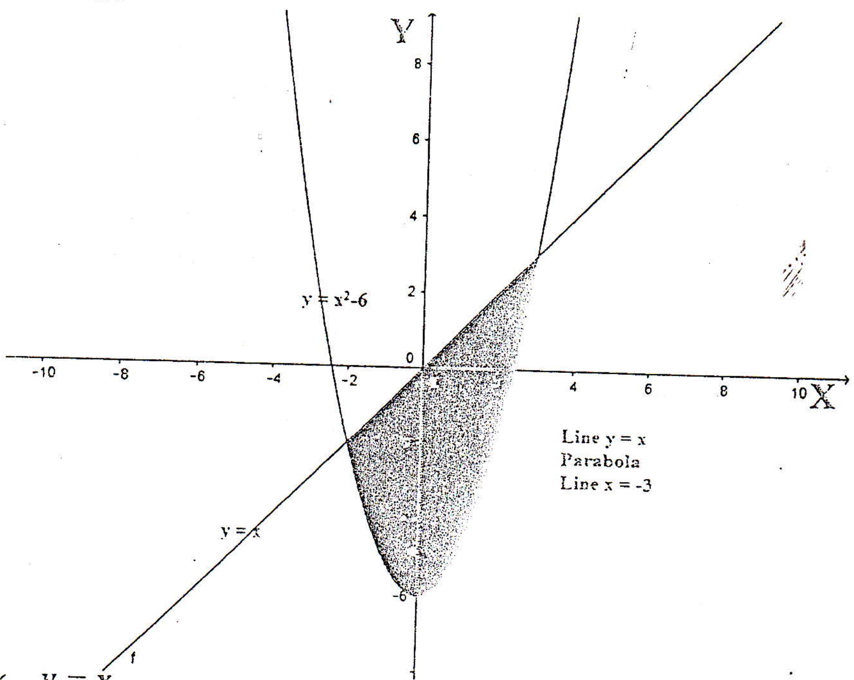
13. Find the value of the complex number $Z = \left[-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right]^{2010}$. Leave your answer in standard form Z = a + bi. 4 marks

Answer:

$$Z = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{2010} = \left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right]^{2010} = \cos\left(2010\frac{2\pi}{3}\right) + i\sin\left(2010\frac{2\pi}{3}\right)$$
$$= \cos(670 \times 2\pi) + i\sin(670 \times 2\pi)$$
$$= 1$$

14. Sketch the plane region bounded by $y = x$ and $y = x^2 - 6$. Then estimate the volume generated by this region when revolved about $x = -3$. 5. 5 marks

Answer:



$$\begin{cases} y = x \\ y = x^2 - 6 \end{cases} \Leftrightarrow x = x^2 - 6 \Rightarrow x^2 - x - 6 = 0 \Rightarrow (x+2)(x-3) = 0 \Rightarrow x = -2 \text{ or } x = 3$$

$$V = 2\pi \int_{-2}^{3} (x+3)(x - x^2 + 6)dx \ (\text{approximate value of the volume})$$

$$V = 2\pi \int_{-2}^{3} (x^3 - 2x^2 + 9x + 18)dx = 2\pi\left[-\frac{x^4}{4} - \frac{2x^3}{3} + \frac{9x^2}{2} + 18x\right]_{-2}^{3} = 2\pi\frac{875}{12}$$

875π

15. The numbers of incorrect answers on a true-false competency test for a random sample of 15 students were recorded as follows: 2, 1, 3, 0, 1, 3, 6, 0, 3, 3, 5, 2, 1, 4 and 2. Find the mean, the median, the mode and the sample standard deviation. 4 marks

Answer:

Frequencies:

Xi	0	1	2	3	4	5	6
Frequency ni	2	3	3	4	1	1	1
Cumulative frequency	2	5	8	12	13	14	15

The mode is 3.

The median 2.

The mean $\bar{x} = \frac{\sum x_i}{15} = \frac{2x0+3x1+3x2+4x3+1x4+1x5+1x6}{15} = \frac{36}{15} = 2.4$

The standard deviation is $\sigma = \sqrt{\frac{\sum(x_i-\bar{x})^2}{N}} = \sqrt{\frac{\sum x_i^2}{N} - \bar{x}} = \sqrt{\frac{126}{15} - 5.76}$

$= \sqrt{8.4 - 5.76} = \sqrt{2.64} \approx 1.6$

SECTION B: Attempt any three questions. (45 marks)

16. Let $f: \Re \rightarrow \Re: f(x) = |x-2| - 1 + \frac{1}{x^2}$

a) Determine the domain of f(x). 2 marks

b) Write $f(x)$ without signs of absolute value. **2 marks**

c) Study the derivability of $f(x)$ at x = -2. **4 marks**

d) Evaluate the limit of $f(x)$-(-x+1) when x approaches -∞ and the limit of $f(x)$-(x-3) when x approaches +∞. Is there any relationship between lines y=x-3 and y=-x+1, and the graph of the function f? 3 marks

e) Evaluate $\int_1^3 f(x)dx$. 4 marks

Answer:

$f(x) = |x-2| - 1 + \frac{1}{x^2}$

a) **Dom f** $= \{x \in IR | x^2 \neq 0\}$
$=]-\infty, 0[\cup]0, +\infty[$

b) For x ∈ Dom f, $f(x) = \begin{cases} x - 3 + \frac{1}{x^2} \; if \; x \geq 2 \\ -x + 1 + \frac{1}{x^2} if \; \leq 2 \end{cases}$

c) For $x \geq 2, f'(x) = 1 - \frac{2}{x^3}$

Thus $f'(2^+) = 1 - \frac{1}{4} = \frac{3}{4}$

For $x \leq 2$, $f'(x) = -1 - \frac{2}{x^3}$

Thus $f'(2^-) = -1 + \frac{1}{4} = -\frac{3}{4}$

Since $f'(2^+) \neq f'(2^-), f'(2) does \; not \; exist$

f is not differentable at x = 2

d) $\lim_{x\to-\infty} [f(x)-(-x+1)] = \lim_{x\to-\infty}\left[\left[-x+1+\frac{1}{x}+\frac{1}{x^2}+x-1\right]\right] = 0$

$\lim_{x\to-\infty} [f(x)-(x-3)] = \lim_{x\to-\infty}\left[\left[x-3+\frac{1}{x}+\frac{1}{x^2}-x+3\right]\right] = 0$

Conclusion: lines $y = -x+1$ and $y = x-3$ are asymptotes to the graph of the function f.

e) $\int_1^3 f(x)dx = \int_1^2\left(-x+1+\frac{1}{x^2}\right)dx + \int_2^3\left(x-3+\frac{1}{x^2}\right)dx$

$= \left[-\frac{x^2}{2}+x-\frac{1}{x}\right]_1^2 \left[-\frac{x^2}{2}+3x-\frac{1}{x}\right]_2^3 = -\frac{1}{3}$

17. a) Solve the equation ${}^{n-1}C_{n-5} = 3\ {}^{n-3}C_{n-1}$ (or $\binom{n-1}{n-5} = 3\binom{n-3}{n-7}$) in the set of positive integers. 6.5 marks

b) Consider the quadratic polynomial z^2-6z+c where c is real. For what values of c does this polynomial have real roots? 2.5 marks

c) Multiply out the expression $(z+7)(z^2-6z+25)$ and hence find all roots (real or complex) of the polynomial $z^3+z^2-17z+175$. 6 marks

Answer:

a) $\binom{n-1}{n-5} = 3\binom{n-3}{n-7}$ condition: $n-7>0$ i.e $n>7$

$\Leftrightarrow \frac{(n-1)!}{(n-5)![(n-1)-(n-5)]!} = 3\frac{(n-3)!}{(n-7)![(n-3)-(n-7)]!}$

$\Leftrightarrow \frac{(n-1)!}{(n-5)!(4)!} = 3\frac{(n-3)!}{(n-7)!(4)!}$

$\Leftrightarrow \frac{(n-1)(n-2)}{(n-5)(n-6)} = 3$

$\Leftrightarrow (n-1)(n-2) = 3(n-5)(n-6)$

$\Leftrightarrow n^2-15n+44 = 0$

$\Leftrightarrow (n-4)(n-11) = 0$

$\Leftrightarrow n = 4$ or $n = 11$

$n=4$ is excluded since less than 7.

The required integer is $n = 11$.

b) The quadratic polynomial z^2-6z+c has real roots if and only if $(-3)^2-c \geq 0$;
That is $c \leq 9$

c) $(z+7)(z^2-6z+25) = z^3 - 6z^2+25z+7z^2-42z+175 = z^3+z^2-17z+175$

Hence $z^3+z^2-17z+175 = 0$

$\Leftrightarrow (z+7)(z^2-6z+25) = 0 \Leftrightarrow z+7 = 0$ or $z^2-6z+25 = 0$

$z+7 = 0 \Leftrightarrow z = -7$

$z^2-6z+25 = 0 \Leftrightarrow z = 3+4i$ or $z = 3-4i$

The roots of the polynomial are $z = -7$, $z = 3+4i$ and $z = 3-4i$

18. a) In Euclidian space, find vector, parametric and symmetric equations for

i) the line through origin and the point (1, 2, 3) 4 marks

ii) The line through (0, 2, -1) and parallel to the line with

parametric equations $\begin{cases} x = 1+2t \\ y = 3t \\ z = 5-7t \end{cases}$

b) Find all cube roots of the complex number $W = -1+i$. Leave your answer in polar form and trigonometric form. 7 marks

Answer:

a) Let O(0, 0, 0) and P(1, 2, 3)

i) The line through origin O and the point P(1, 2, 3) has vector equation $\overrightarrow{OM} = \lambda\overrightarrow{OP}$ where M(x, y, z) is any point on the line.

The parametric equations are $\begin{cases} x = \lambda \\ y = 2\lambda \\ z = 3\lambda \end{cases}$

The symmetric equation is $x = \frac{y}{2} = \frac{z}{3}$

ii) The line through (0, 2, 1) and parallel to the line $\begin{cases} x = 1 + 2t \\ y = 3t \\ z = 5 - 7t \end{cases}$

has parametric equations: $\begin{cases} x = 2t \\ y = 2 + 3t \\ z = -1 - 7t \end{cases}$

Symmetric equation: $\frac{x}{2} = \frac{y-2}{3} = \frac{z+1}{-7}$

Vector equation $\overrightarrow{AM} = \lambda\vec{u}$ where A (0, 2, -1) and $\vec{u}(2, 3, -7)$.

b) W = -1+i

Modulus of W= $\sqrt{1+1} = \sqrt{2}$

$\left.\begin{array}{l} \cos = -\frac{1}{\sqrt{2}} \\ \sin\theta = \frac{1}{\sqrt{2}} \end{array}\right\} \Rightarrow \theta = \frac{3\pi}{4}$

$W = \sqrt{2}x[\mathrm{Cos}(\frac{3\pi}{4}) + i\,sin(\frac{3\pi}{4})]$

Cube roots are of the form $W_k = (\sqrt{2})^{\frac{1}{3}}x[\mathrm{Cos}(\frac{\pi}{4} + \frac{2k\pi}{3}) + i\,sin(\frac{\pi}{4} + \frac{2k\pi}{3})]$

For $k = 0, 1, 2$

Therefore, $W_0 = \sqrt[3]{2}[\mathrm{Cos}(\frac{\pi}{4}) + i\,sin(\frac{\pi}{4})]$

$W_1 = \sqrt[3]{2}[\mathrm{Cos}(\frac{11\pi}{12}) + i\,sin(\frac{11\pi}{12})]$

And $W_2 = \sqrt[3]{2}[\mathrm{Cos}(\frac{19\pi}{12}) + i\,sin(\frac{19\pi}{12})]$

19. a) Compute the sixth degree Taylor polynomial generated by f(x) = lnx about x = 1. Using this result, evaluate $\lim_{x\to 1}\frac{lnx}{x-1}$. 6 marks

b) Find all values of x that satisfy

i) The inequality 2cos(x)+1≥ 0in the interval [0, 2π]; 2.5 marks

ii) the inequality $\log_2\frac{x^2-1}{x+1} = 1$ 3.5 marks

c) If 3 books are picked at random from a shell containing 5 novels, 3books of poems, and one dictionary, what is the probability that

i) the dictionary is selected? 1.5 marks

ii) 2 novels and 1 book of poems are selected? 1.5 marks

Answer:

a) $f(x) = \ln x \Rightarrow f(1) = 0$

$f'(x) = \frac{1}{x} \Rightarrow f'(1) = 1$

$f''(x) = -\frac{1}{x^2} \Rightarrow f''(1) = -1$

$f^3(x) = \frac{2}{x^3} \Rightarrow f^3(1) = 2$

$f^4(x) = -\frac{2x3}{x^4} \Rightarrow f^4(1) = -2x3$

$f^5(x) = \frac{2x3x4}{x^5} \Rightarrow f^5(1) = 2x3x4$

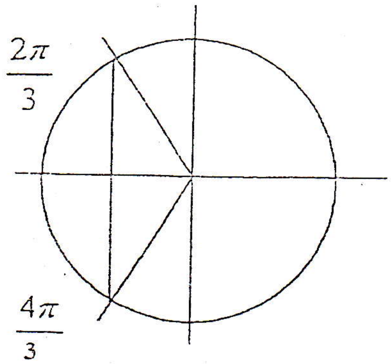
$f^6(x) = -\frac{2x3x4x5}{x^6} \Rightarrow f^6(1) = 2x3x4x5$

The polynomial is

$f(x) = f(1) + f'(1)(x-1) + f''(1)\frac{(x-1)^2}{2!} + f'''(1)\frac{(x-1)^3}{3!} + f^{(4)}(1)\frac{(x-1)^4}{4!} + f^{(5)}(1)\frac{(x-1)^5}{5!} + f^{(6)}(1)\frac{(x-1)^6}{6!}$ Therefore $f(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} - \frac{(x-1)^6}{6}$

$\lim_{x\to 1}\frac{lnx}{x-1} = \lim_{x\to 1}\left[1 - \frac{(x-1)}{2} + \frac{(x-1)^2}{3} - \frac{(x-1)^3}{4} + \frac{(x-1)^4}{5} - \frac{(x-1)^5}{6}\right] = 1$

b) i) $2\cos(x) + 1 \geq 0 \Leftrightarrow \cos(x) \geq -\frac{1}{2}$



$\Leftrightarrow 0 \leq x \leq \frac{2\pi}{3}$ or $\frac{4\pi}{3} \leq x \leq 2\pi$

$S = \left[0, \frac{2\pi}{3}\right] \cup \left[\frac{4\pi}{3}, 2\pi\right]$

ii) $\log_2 \frac{x^2-1}{x+1} = 1 \Leftrightarrow \log_2 \frac{x^2-1}{x+1} = \log_2$, where $\frac{x^2-1}{x+1} > 0$

$\Leftrightarrow \frac{x^2-1}{x+1} = 2 \Leftrightarrow x^2 - 1 = 2(x+1) \Leftrightarrow x^2 - 2x - 3 = 0 \Leftrightarrow (x+1)(x-3) = 0$

$\Leftrightarrow x = -1$ or $x = 3$

$x=-1$ is excluded since the zero of $x+1$

$S = \{3\}$

c) i) p(selecting the dictionary) $= \frac{\binom{5}{2}\binom{3}{2}}{\binom{9}{3}} = \frac{3!6!13!}{9!} = \frac{3x2x13}{9x8x7} = \frac{13}{84}$

ii) p(2 novels and 1 book) $= \frac{\binom{5}{2}\binom{3}{1}}{\binom{9}{3}} = \frac{5!}{2!3!}x\frac{3!}{2!}x\frac{3!6!}{9!} = \frac{5x4x3x2x3}{2x2x9x8x7} = \frac{5}{14}$

20. a) Let *be a binary operation defined on the set Z of all integers by x*y=x-y-3. Determine whether the operation is commutative, and whether there is an identity element. Can you find a symmetric (inverse) of any integer? 8 marksb

b) Sketch and estimate the area of the region bounded by the curves y =x and y = x^2-2.
7 marks

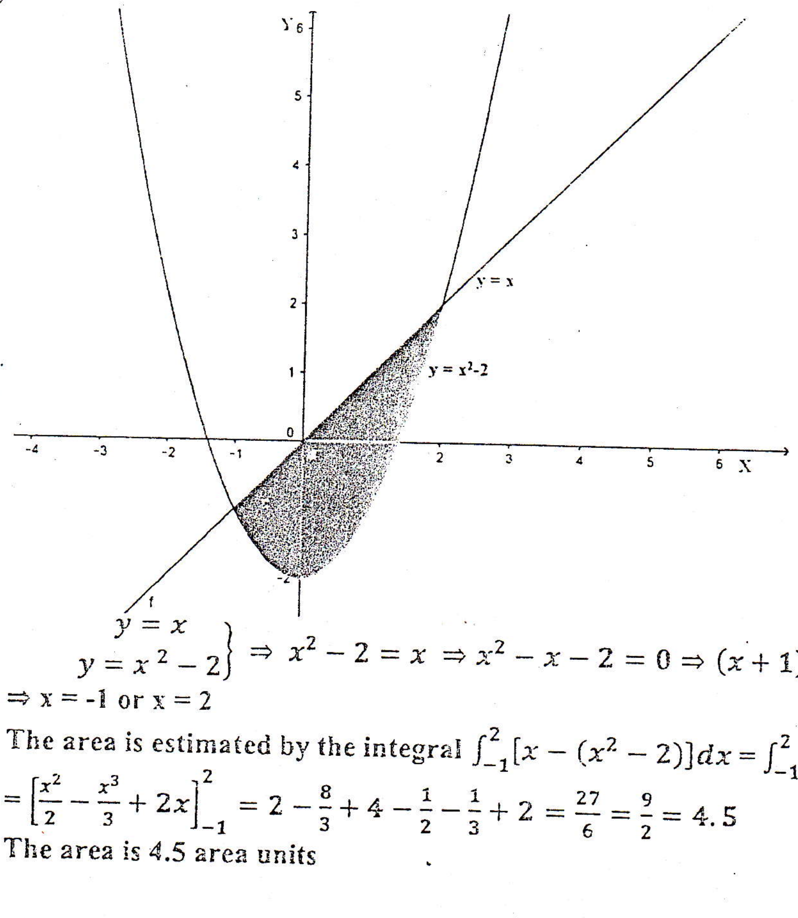
Answer:

a) x*y = x+y+3

The operation is commutative since x*y = x+y+w
= y+x+3(+commutative in Z)
= y*x

Assume that there exisits an identity. Denote by a.
So x*a = x+a+3= x
This equality occurs if a = -3.
So the identity is -3
Let b be a symmetric of x. That means x*b = -3
This occurs if x+b+3=3; that is b= -6-x
So the symmetric of x is -6-x

b)



$$\left.\begin{array}{l} y = x \\ y = x^2 - 2 \end{array}\right\} \Rightarrow x^2 - 2 = x \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x+1)(x-2) = 0$$

⇒ x = -1 or x = 2

The area is estimated by the integral $\int_{-1}^{2}[x-(x^2-2)]dx = \int_{-1}^{2}[x-x^2+2]dx$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3} + 2x\right]_{-1}^{2} = 2 - \frac{8}{3} + 4 - \frac{1}{2} - \frac{1}{3} + 2 = \frac{27}{6} = \frac{9}{2} = 4.5$$

The area is 4.5 area units

ADVENCED LEVEL MATHEMATICS NATIONAL EXAMINATION PAPER 2012
(MCB, MCE, MEG, MPC, MPG, PCM, PEM)

SECTION A: Attempt all questions. (55 marks)

01. Show that C(n-1, p-1) + C(n-1, p) = C(n,p). 4 marks

Answer:

$C(n-1, p-1)+C(n-1, p) = \frac{(n-1)!}{(p-1)!(n-1-(p-1)!} + \frac{(n-1)!}{p!(n-1-p)!} = \frac{(n-1)!}{(p-1)!(n-p)!} + \frac{(n-1)!}{p!(n-p-1)!}$

$= \frac{\frac{n!}{n}}{\frac{p!}{p}(n-p)!} + \frac{\frac{n!}{n}}{p!\frac{(n-p)!}{n-p}} = \frac{n!}{n} x \frac{p}{p!(n-p)!} + \frac{n!}{n} x \frac{n-p}{p!(n-p)!} = \frac{p}{n} x \frac{n!}{p!(n-p)!} + \frac{n-p}{n} x \frac{n!}{p!(n-p)!}$

$= \frac{n!}{p!(n-p)!}\left(\frac{p}{n} + \frac{n-p}{n}\right) = \frac{n!}{p!(n-p)!} x1 = \frac{n!}{p!(n-p)!} = C(n,p) as\ required.$

02. Find the total number of diagonals that can be drawn in a decagon. 3 marks

Answer:

Each diagonal has two end points.

Suppose one has end points A and B, the segment AB and segment BA are the same.

Thus, order is not considered and the combination of 10 points, taken two at a time, is desired.

This gives the total number of line segments. But 10 of them are sides of the polygon.

So the number of diagonals is equal to

$C(10, 2) = \frac{10!}{2!(10-2)!} - 10 = \frac{10!}{2!8!} - 10 = \frac{10x9x8!}{2!8!} = 5x9 - 10 = 35$

03. Determine the continuity of $f(x) = \frac{lnx+tan^{-1}x}{(x-1)(x+1)}$. 3 marks

Answer:

We know that : y = lnx if and only if x>0

y = $\tan^{-1}$x exists if and only if x ∈R

Therefore

y=lnx + $\tan^{-1}$x $\exists \forall x \in (0,+\infty)$

(x-1)(x+1) must be different to zero

$\Rightarrow x \neq 1\ or\ x \neq -1$

Then, the continuity becomes

$x \in (0, 1) \cup (1,+\infty)$

04. Find the value of x if $\sqrt{3}$tanx = 2 sinx 3 marks

Answer:

$\sqrt{3}\tan x = 2 \sin x \Leftrightarrow \sqrt{3}\frac{sinx}{cosx} - 2 \sin x = 0 \Leftrightarrow \sin(\frac{\sqrt{3}}{cosx} - 2) = 0$

$\Leftrightarrow \sin x = 0 \text{ or } \cos x = \frac{\sqrt{3}}{2} \Rightarrow x = k\pi \text{ or } x = \pm\frac{\pi}{6} + 2k\pi, k \in Z$

05. The matrix M(∝) is define by

$M(\propto) = \begin{bmatrix} \cos\propto & \sin\propto \\ -\sin\propto & \cos\propto \end{bmatrix}$.

Verify that $M(\propto)M(\beta) = M(\propto + \beta)$. 2 marks

Answer:

$M(\alpha)M(\beta) = \begin{bmatrix} cos\alpha & sin\alpha \\ -sin\alpha & cos\alpha \end{bmatrix}\begin{bmatrix} cos\beta & sin\beta \\ -sin\beta & cos\beta \end{bmatrix}$

$= \begin{bmatrix} cos\alpha\, cos\beta - sin\alpha\, sin\beta & cos\alpha\, sin\beta + sin\alpha\, cos\beta \\ -sin\alpha\, cos\beta - cos\,\alpha\, sin\beta & -sin\alpha\, sin\beta + cos\alpha\, cos\beta \end{bmatrix}$

$= \begin{bmatrix} cos(\alpha+\beta) & sin(\alpha+\beta) \\ -sin(\alpha+\beta) & cos(\alpha+\beta) \end{bmatrix} = M(\alpha+\beta)\ as\ required$

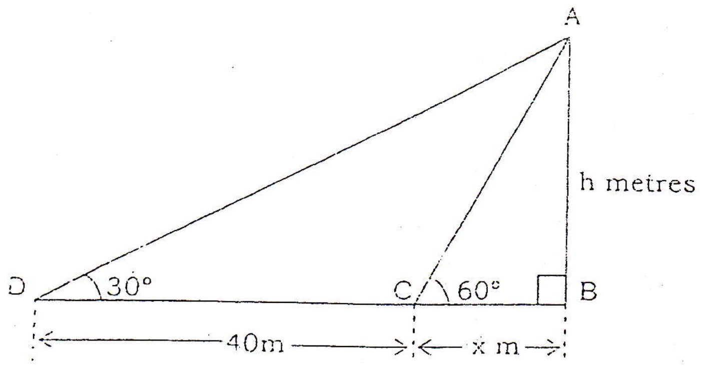
06. A person, standing on the bank of a river observes that the angle subtended by a tree on the opposite bank is 60^0; when he retreats 40 meters from the bank, he finds the angle to be 30^0. Find the breadth of the river and the height of the tree. 5 marks

Answer:

Let: AB= h metres be the height of the tree

CB = x metres be the breadth of the river

So that < BCA = 60^0



Consider D as the second position of the person. Therefore <BDA = 30^0

<ABC = <ABD = 90^0

Therefore: $\frac{AB}{BC} = tan60^0$

i.e. $\frac{h}{x} = tan60^0$

$\Leftrightarrow \frac{h}{x} = \sqrt{3} \Rightarrow h = \sqrt{3}x$ (1)

$\frac{AB}{BD} = tan30^0$

$\Leftrightarrow \frac{h}{40+x} = \frac{1}{\sqrt{3}} \Leftrightarrow h\sqrt{3} = 40 + x$ (2)

Putting (1) into (2), we get

$\sqrt{3}.\sqrt{3}x = 40 + x \Leftrightarrow 3x=40+x \Leftrightarrow 2x=40 \Leftrightarrow x=20$

$Then\ h = 20\sqrt{3}\ metres \approx 34.6\ metres$

Hence the height of tree is 34.6 metres and breath of river is equal to 20 metres.

07. If T_p, T_q and T_r are p^{th}, q^{th} and r^{th} terms of an arithmetic progression, then find the value of $\begin{vmatrix} T_p & T_q & T_r \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$ 3 marks

Answer:

Let 'a' be the first term and 'd' the common difference.

Therefore T_p=a+(p-1)d

T_q= a+(q-1)d

T_r=a+(r-1)d

Therefore $\Delta = \begin{vmatrix} a+(p-1)d & a+(q-1)d & a+(r-1)d \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$

Applying $C_2 \rightarrow C_2 - C_1$, $C_3 \rightarrow C_3 - C_1$

Let $(x-2)^2+(y+1)^2+(z-3)^2=R^2$

Since T is a point of the sphere, we must have:

$(1-2)^2+(2+1)^2+(-3-3)^2 = R^2 \Leftrightarrow R^2 = 46$

Therefore: $(x-2)^2+(y+1)^2+(z-3)^2 = 46$

Or $x^2+y^2+z^2-4x+2y-6z-32 = 0$

Let $\vec{u}(\alpha,\beta,\gamma)$ *be a vector direction of the line. We must get* $\vec{u} \perp \overrightarrow{MT}$,

i.e : $\vec{u} \perp \overrightarrow{MT} = 0$

Thus $-\alpha + 3\beta - 6\gamma = 0$

This equation has many solutions

Let $\beta = \frac{\alpha}{3} + 2\gamma$

The system of parametric equations has the form

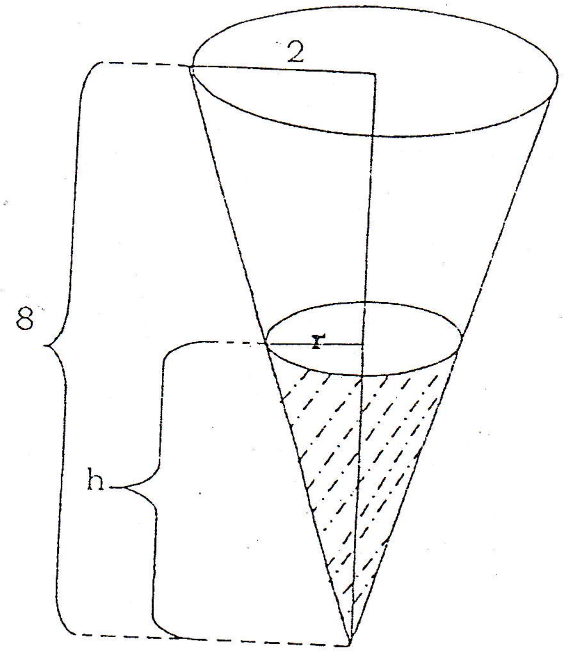
$$\begin{cases} x - 1 = \alpha t \\ y - 2 = \left(\frac{\alpha}{3} + 2\gamma\right) t \\ z + 3 = \gamma t \end{cases}$$

The system represented family lines included in the plane tangent to the sphere.

12. A tank is the form of an inverted cone having height 8 meters and radius 2 meters. Water is flowing into the tank at the rate of $\frac{1}{8}\ m^3/minute$. How fast is the water level rising when the water is 2.5 meters deep? 4 marks

Answer:

Let h = height and r = base radius of water at time t.



$\frac{r}{h} = \frac{2}{8} \Leftrightarrow r = \frac{h}{4}$

Therefore V = volume of water at time t.

$\Leftrightarrow V = \frac{\pi r^2 h}{3} = \frac{\pi}{3}\left(\frac{h}{4}\right)^2 h = \frac{\pi h^3}{48}$

Therefore $\frac{dv}{dt} = \frac{\pi}{48} x 3h^2 x \frac{dh}{dt} = \frac{\pi}{16} h^2 \frac{dh}{dt}$

When $\frac{dv}{dt} = \frac{1}{8}$ and $h = 2.5$

We obtain $\frac{1}{8} = \frac{\pi}{16}(2.5)^2 \frac{dh}{dt}$

Therefore $\frac{dh}{dt} = \frac{8}{25\pi}$

Then, the water level is rising at the rate of $\frac{8}{25\pi}$ $m/minute$

13. Calculate:

a) $\int \frac{sinx}{1+sinx} dx$

b) $\int_0^2 \frac{5x++1}{x^2+4} dx$

Answer:

a) $\int \frac{sinx}{1+sinx} dx = \int \frac{1+sinx-1}{1+sinx} dx = \int (1 - \frac{1}{1+sinx}) dx = \int dx - \int \frac{1}{1+sinx} dx = x - I_1$

$I_1 = \int \frac{1}{1+sinx} dx = \int (\frac{1}{1+sinx} x \frac{1-sinx}{1-sinx}) dx = \int \frac{1-sinx}{cos^2x} dx = \int \left(\frac{1}{cos^2x} - \frac{sinx}{cos^2x}\right) dx$

$= \int (sec^2x - \frac{1}{cosx} x \frac{sinx}{cosx}) dx == \int sec^2x dx - \int secx\, tanx\, dx = tanx - secx + C$

$I = x - tanx + secx + c$

b) $\int_0^2 \frac{5x++1}{x^2+4} dx = \int_0^2 \frac{5xdx}{x^2+4} + \int_0^2 \frac{dx}{x^2+4} = \frac{5}{2}\int_0^2 \frac{2xdx}{x^2+4} + \int_0^2 \frac{dx}{x^2+4} = \left[\frac{5}{2} ln|x^2+4| + \frac{1}{2} tan^{-1}\frac{x}{2}\right]_0^2$

$= \left(\frac{5}{2} ln8 + \frac{1}{2} tan^{-1} 1\right) - \left(\frac{5}{2} ln4 + \frac{1}{2} tan^{-1} 0\right) = \left(\frac{5}{2} ln8 + \frac{1}{2} x \frac{\pi}{4}\right) - \left(\frac{5}{2} ln4 + \frac{1}{2} x0\right)$

$= \frac{5}{2}(ln8 - ln4) + \frac{\pi}{8} = \frac{5}{2} ln\frac{8}{4} + \frac{\pi}{8} = \frac{5}{2} ln2 + \frac{\pi}{8}$

14. a) In a single throw of two dice, determine the probability of getting a total of 2 or 4. **2 marks**

b) The letters of the word "DIVORCE" are arranged at random. Find the probability that the vowels may occupy the even places. **2 marks**

Answer:

a) Two dice can be thrown in 6x6 ways

Here S = {(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),.....(6,5),(6,6)}

Let A be event of getting a total of 2 or 4.

Therefore A = {(1,1),(1,3),(2,2),(3,1))}

Required probability becomes

$\frac{n(A)}{n(S)} = \frac{4}{36} = \frac{1}{9}$

b) The word "DIVORCE" has 7 letters. They can be arranged among themselves in (7!) ways. In this word "DIVORCE" there is 3 vowels and 4 consonants. These 3 vowels have to be placed in three even places: 2nd, 4th and 6th; and they can occupy these 3 places in (3!) ways and 4 consonants can occupy the remaining 4 places in (4!) ways.

Thus the number of ways favorable to the event is $3! x4!$

Then, required probability becomes $\frac{3!4!}{7!} = \frac{3x2x1x4!}{7x6x5x4!} = \frac{1}{7x5} = \frac{1}{35}$

15. Find the sum of $1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \ldots$ 3 marks

Answer:

$S = 1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+ \ldots$

We know that $1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+ \ldots\ldots = e^x$

Putting $x = \pm 1$,

We get: $1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\cdots = e$ (1)

And: $1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}+\cdots = e^{-1}$ (2)

(1) + (2) give us: $2(1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+ \ldots) = e+e^{-1}$, Then $1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+ \ldots = \frac{e+e^{-1}}{2}$

SECTION B: Attempt ONLY THREE questions (45 marks)

16. Consider a real valued numerical function defined as f: IR → IF

$x \to \frac{1}{2}x^2e^{x+1}$.

a) Find the domain of function f(x) 1 mark
b) Find the intersection with axis of coordinates. 2 marks
c) Find the asymptotes 5 marks
d) Discuss the first and second derivative of f(x) 3 marks
e) Sketch the graph of f(x) 2 marks

Answer:

a) $f(x) = \frac{1}{2}x^2e^{x+1}$

Dom f = IR

b) **Intersection with axis of coordinates:**

When $x = 0$, $y = \frac{1}{2}x0^2xe^{0+1} = 0$

$\forall x \in IR / \frac{1}{2}x^2e^{x+1} > 0$ *except when* $x = 0 \Rightarrow f(x) = 0$

The intersection with axis of coordinates in the point $(0,0)$.

c) Asymptotes

i) Horizontal asymptotes;

$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x^2e^{x+1}}{2} = +\infty$

$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^2}{2}.e^{x+1} = \infty.0\ I.F$

$f(x) = \frac{\frac{x^2}{2}}{e^{-(x+1)}} \Rightarrow \lim_{x\to-\infty} \frac{\frac{x^2}{2}}{e^{-(x+1)}} = \frac{\infty}{\infty}\ I.R$

Using Hospital's rule, we obtain;

$\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} \frac{(\frac{1}{2}x^2)}{(e^{-(x+1)})} = \lim_{x\to-\infty} \frac{x}{(-e^{-(x+1)})} = \frac{-\infty}{-\infty}\ I.F$

Again $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x'}{(-e^{-(x+1)})'} = \lim_{x\to-\infty} \frac{1}{(e^{-(x+1)})} = 0$

Then, horizontal asymptote $\equiv y = 0$

ii) Vertical asymptote cannot exist because dom f(x) = IR.

iii) Oblique asymptote

$a = \lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{1}{2}xe^{x+1} = +\infty$

On the other hand

$a = \lim\limits_{x \to -\infty} \frac{f(x)}{x} = \lim\limits_{x \to -\infty} \frac{1}{2} x e^{x+1} = 0$

Then, oblique asymptote does not exist.

d) $f'(x) = \frac{1}{2} . 2xe^{x+1} + \frac{1}{2} x^2 . e^{x+1} . 1 = xe^{x+1} + \frac{x^2 e^{x+1}}{2} = xe^{x+1}(1 + \frac{x}{2})$

$f'(x) = 0 \Leftrightarrow = x(1 + \frac{x}{2}) = 0 \Leftrightarrow x = 0$ or $x = -2$

When $x = 0$, $y = 0$.

When $x = -2$, $y = \frac{1}{2}(-2)^2 e^{-2+1} = \frac{2}{e}$

The turning points are $(0,0)$ and $(-2, 2e^{-1})$.

$f''(x) = \left(xe^{x+1} + \frac{x^2}{2} e^{x+1}\right) = e^{x+1} + xe^{x+1} + xe^{x+1} + \frac{x^2}{2} e^{x+1}$

$= e^{x+1} + 2xe^{x+1} + \frac{x^2}{2} e^{x+1} = e^{x+1}(1 + 2x + \frac{x^2}{2})$

$f''(x) = 0 \Leftrightarrow 1 + 2x + \frac{x^2}{2} = 0 \ (\forall x \in IR, e^{x+1} > 0)$

$x^2 + 4x + 2 = 0$

$\Delta = 16 - 8 = 8$

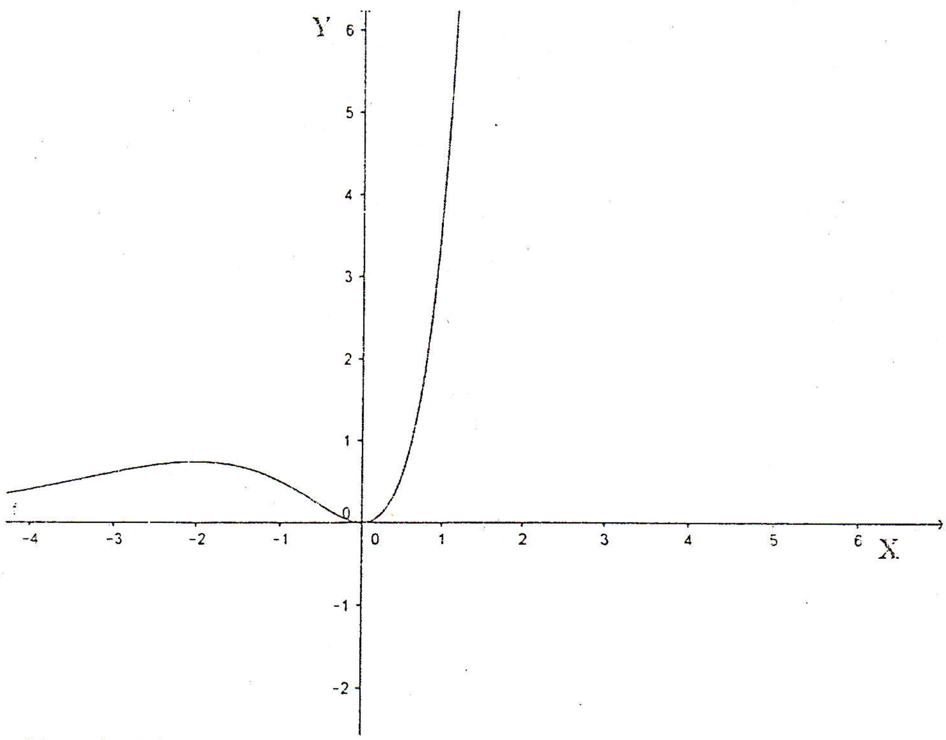
$x_1 = \frac{-4+\sqrt{8}}{2} = -2 + \sqrt{2}$

$x_2 = \frac{-4-\sqrt{8}}{2} = -2 - \sqrt{2}$

Variable table

x	$-\infty$ $-2-\sqrt{2}$ -2 $-2-\sqrt{2}$ 0 $+\infty$
f'(x)	+ + + + 0 - - - - - - - 0 + + + +
f''(x)	+ + + + + + + 0 - - - - - - - 0 + + + + + +
f(x) Concavity	$M = 2e^{-1}$ ↗ ↘ ↗ m=0

e) Graph of f(x)



17. The sides of perfect die are colored as follows: three sides are orange, two sides are green and one side is red. A player bets 200 RWF is refunded for each throw. When red face of the die is up, a player is refunded 10% of 200 RWF, when orange face is up, a player is refunded 30% of 200 RWF and when green face is up, a player is given 500 RWF. If X is the difference between the refunded money and the betted money,
    a) Determine the sets of values of x and the distribution probability of X. **5.5 marks**
    b) Calculate the mathematical expectation E(X) of X and interpret the obtained values. 4 marks
    c) Calculate the variance and the standard deviation of X. **5.5 marks**

Answer:

a) $\Omega = \{red\,face, orange\,face, green\,face\}$

   10% of 200F = 20F
   30% of 200F = 60F
   Let X: X = 20F-200F =-180F
   X= 60F-200F = -140F
   X= 500F-200F = 300F
   $\Rightarrow X(\Omega) = \{-180, -140, 300\}$
   Therefore $P(X=-180) = \frac{1}{6}$
   $P(X=-140) = \frac{3}{6} = \frac{1}{2}$
   $P(X=300) = \frac{2}{6} = \frac{1}{3}$

Distribution probability in tabular form:

X_i	-180	-140	300
$P(X_i)$	1/6	½	1/3

b) $E(x) = \sum_{i=1}^{n} P(X_i)X_i = \frac{1}{6}(-180) + \frac{1}{2}(-140) + \frac{1}{3}(300) = -30 - 70 + 100 = 0$

Interpretation of E(X) = 0:

The game is balanced; the player is equally likely to lose or gain. After many trials, the player will neither lose or gain.

c) $V(X) = \sum_{i=1}^{n} P(X_i)X_i^2 - E(X)^2 = \sum_{i=1}^{n} P(X_i)X_i^2 - 0 = \sum_{i=1}^{n} P(X_i)X_i^2 =$

$-\frac{1}{6}(-180)^2 + \frac{1}{2}(-140)^2 + \frac{1}{3}(300)^2 = 5400 + 9800 + 30\,000 = 45\,200$

And

$\rho_x = \sqrt{V(X)} = \sqrt{45\,200} = 212.60$

18. A straight line passes through points A(-1, -5), B(0, -8) and $2y + 16 = 4x^2$ is the equation of the curve C.

a) Find the equation on the straight line AB. **1 mark**

b) In the same Cartesian plane, draw the straight line AB and the curve C. **3 marks**

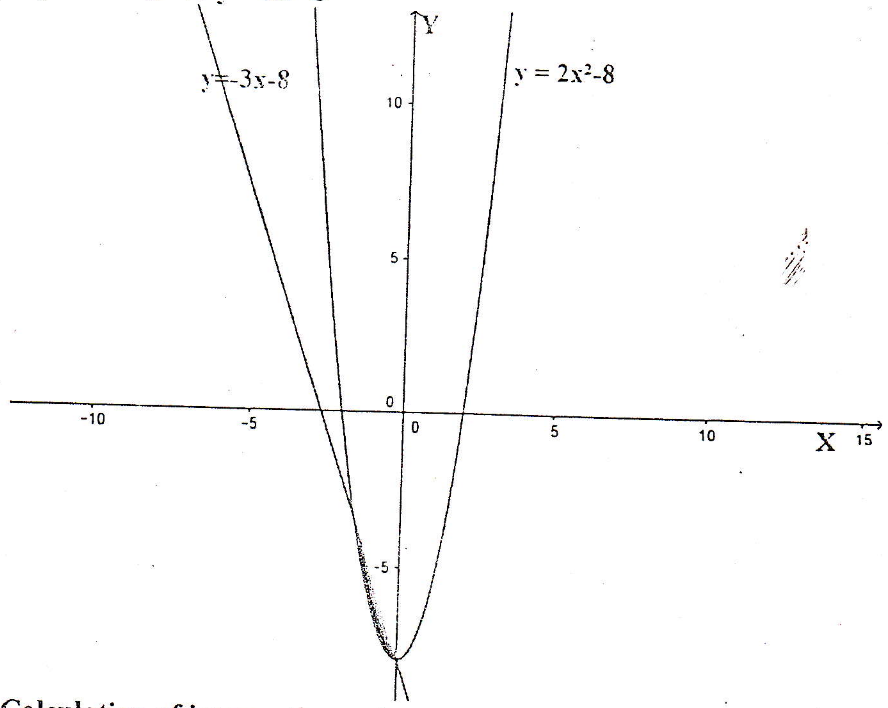
c) Calculate the area between the curve C and the straight line AB. **6 marks**

d) Calculate the volume of solid of revolution about the x-axis of the surface area in c) above. **5 marks**

Answer:

a) $AB \equiv y + 5 = \frac{-8=5}{1}(x + 1) \Leftrightarrow AB \equiv y = -3x - 8$

b) $2y+16 = 4x^2 \Leftrightarrow y = 2x^2-8$



c) **Calculation of intersection points**

$y = -3x - 8$ and $y = 2x^2 - 8 \Leftrightarrow 2x^2 + 3x = 0 \Leftrightarrow x(2x+3) = 0 \Leftrightarrow x = 0$ or $x = -3/2$

Surface area $A = \left|\int_{-\frac{3}{2}}^{0}[(2x^2 - 8) - (-3x - 8)dx\right| = \left|\int_{-\frac{3}{2}}^{0}[(2x^2 + 3x)dx\right|$

$= \left|\left[\frac{2x^3}{3} + \frac{3x^2}{2}\right]_{-3/2}^{0}\right| = \left|0 - \left[\frac{2}{3}\left(\frac{-27}{8}\right) + \frac{3}{2}\left(\frac{9}{4}\right)\right|\right| = \left|0 + \frac{9}{4} - \frac{27}{8}\right| = \left|\frac{18-27}{8}\right| = \frac{9}{8}$ *Sq units*

d) The volume of solid of revolution V:

$$V = \pi\int_{-\frac{3}{2}}^{0}[(2x^2-8)^2-(-3x-8)]dx = \int_{-\frac{3}{2}}^{0}(4x^4-41x^2-48x)dx$$

$$=\pi\left[\frac{4x^5}{5}-\frac{41x^3}{3}-\frac{48x^2}{2}\right]_{-3/2}^{0} = \pi\left\{0-\left[\frac{4}{5}\left(\frac{-243}{32}\right)-\frac{41}{3}\left(\frac{-27}{-8}\right)-24\left(\frac{9}{4}\right)\right]\right\}$$

$$=\pi\left[\frac{243}{40}-\frac{369}{8}+\frac{216}{4}\right] = \pi\left(\frac{243-1845+2160}{40}\right) = \frac{558\pi}{40}\ Cubic\ units$$

19. a) Suppose f and g are linear transformations on real vector space IR^2 with their respective representative matrices $F = \begin{bmatrix}1 & -1\\0 & 2\end{bmatrix}$ and $G = \begin{bmatrix}3 & 0\\-2 & 1\end{bmatrix}$ relative to the basis B. Find the matrix that represents gof.

b) Find a vector u such that f(u) = 2u and vector v such that f(v) = v 4 marks

c) Prove that B = (u, v) is a basis of the vector space IR^2 2 marks

d) Write the matrix T that represents f relative to the basis B. 4 marks

e) Find a relationship between F and T. 2 marks

Answer:

a) $F = \begin{bmatrix}1 & -1\\0 & 2\end{bmatrix}$ and $G = \begin{bmatrix}3 & 0\\-2 & 1\end{bmatrix}$

The matrix representing g of f is

$$G.F = \begin{bmatrix}3 & 0\\-2 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\0 & 2\end{bmatrix} = \begin{bmatrix}3 & -3\\-2 & 4\end{bmatrix}$$

b) $f(u) = 2u \Leftrightarrow \begin{bmatrix}1 & -1\\0 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}2x\\2y\end{bmatrix}$

When $u = \begin{bmatrix}x\\y\end{bmatrix}$

$$\Leftrightarrow \begin{cases}x-y=2x\\2y=2y\end{cases} \Leftrightarrow \begin{cases}y\in IR\\x=-y\end{cases}$$

Therefore u is a vector of the form $\binom{-y}{y} = y\binom{-1}{1}$, y ∈ IR

Take for example: $u = \binom{-1}{1}$

$f(v) = v \Leftrightarrow \begin{bmatrix}1 & -1\\0 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix}$

with $v = \begin{bmatrix}x\\y\end{bmatrix}$

$$\Leftrightarrow \begin{cases}x-y=x\\2y=y\end{cases} \Leftrightarrow \begin{cases}x\in IR\\y=0\end{cases}$$

V is of the form $\begin{bmatrix}x\\0\end{bmatrix} = x\begin{bmatrix}1\\0\end{bmatrix}, x\in IR$

c) $B = (u, v) = \left(\binom{-1}{1}, \binom{1}{0}\right)$ *is a basis of* IR^2

Since $\begin{vmatrix}-1 & 1\\1 & 0\end{vmatrix} = -1 \neq 0$

And B having two vectors, is a generator system in as much as u and v are linearly independent.

d) Let T be the matrix that represents f in B.

We must have $\begin{cases}f(u)=2u\\f(v)=v\end{cases} \Leftrightarrow \begin{cases}f(u)=2u+0v\\f(v)=0u+v\end{cases}$

Therefore $T = \begin{vmatrix}-1 & 1\\1 & 0\end{vmatrix}$